COURSE INFORMATION SHEET

Course Code: **ME24303**
Course Title: **Mechanical Vibration**
Pre-requisite(s): **Basics of Mechanical Engineering**
Co- requisite(s): **NIL**
Credits: **3** **(**L:3 T:0 P:0)
Class schedule per week: **3**
Class: **B. Tech**
Semester / Level: **FIFTH**
Branch: **Mechanical Engineering**
Name of Teacher:

COURSE OBJECTIVES

This course envisions to impart to students to:

1.	Obtain the model of vibratory systems of single, multi degrees of freedom system as well as continuous systems.
2.	Perform modal analysis of different systems.
3.	Understand the experimental procedures in vibration analysis.

COURSE OUTCOMES (COs)

After the completion of this course, students will be able to:

CO1	Explain the basic elements of vibration models
CO2	Analyze multi degrees of freedom systems.
CO3	Solve the natural frequencies and mode shapes of a vibrating system.
CO4	Analyse the vibration of continuous systems.
CO5	Apply the knowledge of vibration in practical problems.

SYLLABUS

MODULE	(NO. OF LECTURE HOURS)
Module – I Equation of motion with Newton's method, D' Alembart principle, and energy method, natural frequency, response due to initial condition, Viscously damped free vibration, overdamped, critically damped, and underdamped system, damped frequency, logarithmic decrement, coulomb damping.	**8**
Module – II Harmonic excitation of undamped system, resonance, beating phenomenon, Harmonic excitation of damped system, Magnification factor, rotating unbalance, support excitation, vibration isolation, whirling of rotating shaft.	**8**
Module – III Free vibration analysis of two degrees of freedom system, derivation of equation of motion, coordinate coupling, principal coordinates, orthogonality of modes, Lagrange's equation, Forced vibration analysis of two degrees of freedom system due to harmonic excitation; torsional vibration with two rotor masses.	**8**
Module – IV Determination of natural frequencies and mode shapes of multi degree of freedom system matrix formulation, influence coefficient, flexibility matrix, stiffness matrix, numerical methods: Dunkerley's method, and Method of matrix iteration.	**8**
Module – V Introduction to the vibration of continuous systems; Lateral vibration of string; Longitudinal vibration of bar; Torsional vibration of uniform shaft; Transverse vibration of beams having different types of supports (Euler- Bernoulli beam).	**8**

TEXTBOOKS:

T1. Theory of Vibration with Applications: W. T. Thomson and Marie Dillon Dahleh, Pearson Education.

T2. Introductory Course on Theory and practice of Mechanical Vibrations by J.S.Rao and K. Gupta, Wiley Eastern Ltd

REFERENCE BOOKS:

R1. Mechanical vibrations by Thammaiah Gowda, Jagadeesha T and D V Girish, McGraw Hill.

GAPS IN THE SYLLABUS (TO MEET INDUSTRY/PROFESSION REQUIREMENTS)

Real-world case studies, condition monitoring techniques, and industry-standard practices for vibration analysis and control.

POS MET THROUGH GAPS IN THE SYLLABUS: PO 1-5.

TOPICS BEYOND SYLLABUS/ADVANCED TOPICS/DESIGN

Nonlinear vibration analysis and active vibration control techniques.

POS MET THROUGH TOPICS BEYOND SYLLABUS/ADVANCED TOPICS/DESIGN: PO 1-5, PO 11-12.

COURSE OUTCOME (CO) ATTAINMENT ASSESSMENT TOOLS & EVALUATION PROCEDURE

DIRECT ASSESSMENT

Assessment Tool	% Contribution during CO Assessment
Progressive Evaluation	50
End Semester Examination	50

Continuous Internal Assessment	% Distribution
Mid Semester Examination	25
Quiz, Assignment	10 + 10
Teacher's Assessment	5

Assessment Components	CO1	CO2	CO3	CO4	CO5
Continuous Internal Assessment	√	√	√	√	√
Semester End Examination	√	√	√	√	√

INDIRECT ASSESSMENT

1. **Student Feedback on Course Outcome**

COURSE DELIVERY METHODS

CD1	Lecture by use of boards/LCD projectors/OHP projectors	√
CD2	Assignments/Seminars	√
CD3	Laboratory experiments/teaching aids	
CD4	Industrial/guest lectures	
CD5	Industrial visits/in-plant training	
CD6	Self- learning such as use of NPTEL materials and internets	√
CD7	Simulation	

MAPPING BETWEEN COURSE OUTCOMES AND POs and PSOs

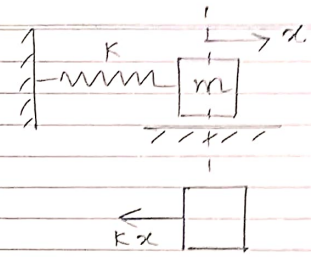
	PO 1	PO 2	PO 3	PO 4	PO 5	PO 6	PO 7	PO 8	PO 9	PO 10	PO 11	PO 12	PSO 1	PSO 2	PSO 3
CO1	3	2	1	2	1		2	-	1	1		2	3	2	1
CO2	3	3	2	2	3	1	2	1	2	2	1	2	3	3	2
CO3	3	3	3	3	3	2	3	1	2	3	2	3	3	3	3
CO4	3	2	3	3	2	3	3	2	3	3	3	3	3	3	3
CO5	3	2	3	3	2	3	3	2	3	3	3	3	3	3	3

Grading: No correlation – 0, Low correlation - 1, Moderate correlation – 2, High Correlation - 3

MAPPING BETWEEN COURSE OUTCOMES AND COURSE DELIVERY METHOD

Course Outcomes	Course Delivery Method
CO1	CD1, CD2, CD 6
CO2	CD1, CD2, CD 6
CO3	CD1, CD2, CD 6
CO4	CD1, CD2, CD 6
CO5	CD1, CD2, CD 6

SL. NO.	DATE	TITLE	PAGE NO.	TEACHER'S SIGNATURE



Eqⁿ of motion : $m\ddot{x} = -Kx$

where $\ddot{x} = \frac{d^2x}{dt^2}$, $\dot{x} = \frac{dx}{dt}$

Eqⁿ of Equilibrium : $\boxed{m\ddot{x} + Kx = 0}$

$$\ddot{x} + \frac{K}{m}x = 0$$

$$D^2 + \frac{K}{m} = 0$$

$$D = \pm i\sqrt{\frac{K}{m}}$$

$$x = A_1 e^{i\sqrt{\frac{K}{m}}t} + A_2 e^{-i\sqrt{\frac{K}{m}}t}$$

To have real value of x,

$$A_1 = A_2$$

$$A_1 = A_3(\cos\phi - i\sin\phi) = A_3 e^{i\phi}$$

$$A_2 = A_3(\cos\phi + i\sin\phi) = A_3 e^{-i\phi}$$

$$i\sqrt{\frac{K}{m}}t + i\phi \qquad -i\sqrt{\frac{K}{m}}t$$

$$x = A_3 e^{i\left(\sqrt{\frac{K}{m}}t+\phi\right)} + A_3 e^{-i\left(\sqrt{\frac{K}{m}}t+\phi\right)}$$

$$= A_3 e^{\cdot} + A_3 e$$

$$= A\cos\left(\sqrt{\frac{K}{m}}t + \phi\right)$$

Time Period $= \dfrac{2\pi}{\sqrt{\frac{K}{m}}}$

ang. freq $= \dfrac{2\pi}{T} = \omega_n = \sqrt{\dfrac{K}{m}}$

Let initial conditions are :-

at $t=0$

$$x(t)\Big|_{t=0} = x_0$$

$$\frac{dx}{dt}\Big|_{t=0} = v_0$$

$$x_0 = A\cos\phi$$

$$\cos\phi = \frac{x_0}{A} \quad \text{——— (1)}$$

$$\frac{dx}{dt} = -A\sin\left(\sqrt{\frac{K}{m}}t + \phi\right)\sqrt{\frac{K}{m}}$$

$$v_0 = -A\sin\phi\sqrt{\frac{K}{m}}$$

$$\sin\phi = \frac{-v_0}{A}\sqrt{\frac{m}{K}} \quad \text{——— (2)}$$

(2)/(1),

$$\boxed{\tan\phi = \frac{-v_0}{x_0}\sqrt{\frac{m}{K}}}, \quad \tan\phi = \frac{-v_0}{x_0\omega_n}$$

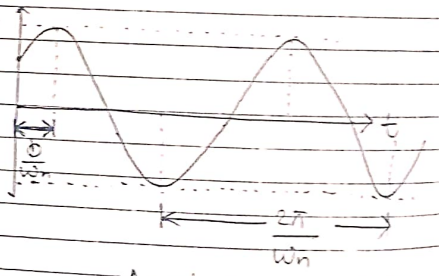
$(1)^2 + (2)^2$

$$\left(\frac{x_0}{A}\right)^2 + \left(\frac{v_0}{A}\right)^2\frac{1}{\omega_n} = 1$$

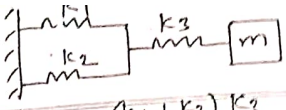
$$\boxed{A = \sqrt{x_0^2 + \frac{v_0^2}{\omega_n^2}}}$$

$$x = \sqrt{x_0^2 + \frac{v_0^2}{\omega_n^2}}\cos\left(\omega_n t + \phi\right)$$

$$\phi = \tan^{-1}\left(\frac{v_0}{x_0\omega_n}\right), \quad \omega_n = \sqrt{\frac{k}{m}}$$



time $= \dfrac{\text{Angular distance}}{\text{Angular frequency}}$



$$k_{eq} = \frac{(k_1 + k_2)\,k_3}{(k_1 + k_2) + k_3}$$

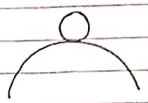
Q → How a system behaves if disturbed slightly from equilibrium?

⇒ The system oscillates about the equilibrium without executing a secular trend. neither trending to the equilibrium nor greatly departing from it. so that the motion remains bounded.



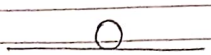
Stable equilibrium

⇒ The system trends away from equilibrium secularly. In this case the equilibrium is unstable



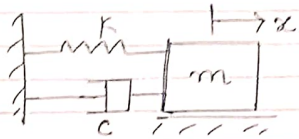
Unstable equilibrium

⇒

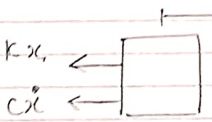


Neutral equilibrium

Viscous Damping of Single Degree of Freedom (SDOF) system



Start with free body diagram



where,

$\ddot{x} = \frac{d^2x}{dt^2}$

$\dot{x} = \frac{dx}{dt}$

$m\ddot{x} = -kx - c\dot{x}$

Inertia Force | Spring Force | Damping Force

$m\ddot{x} + kx + c\dot{x} = 0$ (Equilibrium equation

$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$

$\frac{k}{m} = \omega_n^2$ $\quad \frac{c}{m} = 2\zeta\omega_n$ [zeta ζ]

{ c = Coeff. of damping
ζ (zeta) = Damping Factor }

$\tau = \mu \frac{dv}{dh}$

$f = \left(\frac{\mu A}{h}\right) v$

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = 0$$

Assuming soln of the diff. eqn is

$$x = e^{st}$$
$$\dot{x} = s e^{st}$$
$$\ddot{x} = s^2 e^{st}$$

Put back in the diff^n eq^n:

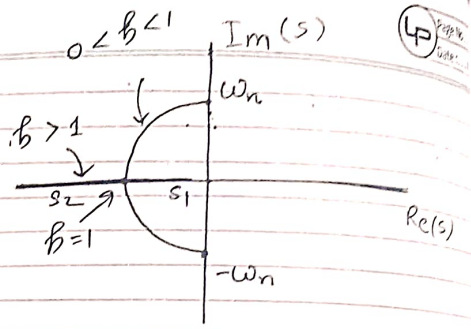
$$s^2 e^{st} + 2\zeta\omega_n s e^{st} + \omega_n^2 e^{st} = 0$$

$$e^{st}\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right) = 0$$

$$s = \left(-\zeta \pm \sqrt{1-\zeta^2}\right)\omega_n$$

$$s_1 = -\zeta\omega_n + i\sqrt{1-\zeta^2}\,\omega_n$$
$$s_2 = -\zeta\omega_n - i\sqrt{1-\zeta^2}\,\omega_n$$

$$x = A_1 e^{\left(-\zeta\omega_n + i\sqrt{1-\zeta^2}\,\omega_n\right)t} + A_2 e^{\left(-\zeta\omega_n - i\sqrt{1-\zeta^2}\,\omega_n\right)t}$$



If $\zeta = 0$

$$s_1 = i\omega_n$$
$$s_2 = -i\omega_n$$

$$s_1 = -\zeta\omega_n + \sqrt{\zeta^2 - 1}\,\omega_n$$
$$s_2 = -\zeta\omega_n - \sqrt{\zeta^2 - 1}\,\omega_n$$

If $\zeta \uparrow$ then $s_1 \to 0$, $s_2 \to -\infty$

(I) $\zeta > 1$ overdamping

(II) $\zeta = 1$ Critical damping

(III) $\zeta < 1$ Under damping

$\zeta > 1$

(I) Soln is

$$x(t) = e^{-\zeta\omega_n t}\left[A_1 e^{\sqrt{\zeta^2-1}\,\omega_n t} + A_2 e^{-\sqrt{\zeta^2-1}\,\omega_n t}\right]$$

In case

$\zeta = 1$

We have repeated roots

$s_1 = -\omega_n$

$s_2 = -\omega_n$

The soln can be shown as

$$x(t) = (A_1 + tA_2)\, e^{-\omega_n t}$$

Let initial conditions are,

$$x(t)\Big|_{t=0} = x_0 \quad , \quad \frac{dx(t)}{dt}\Big|_{t=0} = v_0$$

$$\boxed{A_1 = x_0}$$

$$\frac{dx}{dt} = A_1 e^{-\omega_n t}(-\omega_n) + A_2\left[1.e^{-\omega_n t} + e^{-\omega_n t}.(-\omega_n)t\right]$$

$$v_0 = -A_1\omega_n + A_2$$

$$\boxed{A_2 = v_0 + \omega_n x_0}$$

$$e^{-\omega_n t}\left[-A_1\omega_n + A_2(1-\omega_n t)\right] =$$

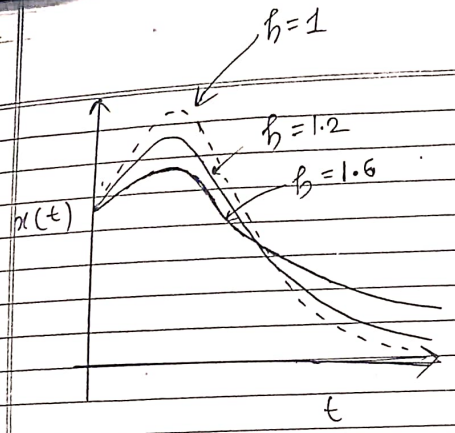
$$A_1\omega_n = A_2(1-\omega_n t)$$

$$A_1\omega_n = A_2 - A_2\omega_n t$$

$$t = \frac{A_2 - A_1\omega_n}{A_2\omega_n}$$

$$\frac{v_0 + \omega_n x_0 - x_0\omega_n}{(v_0 + \omega_n x_0)\omega_n}$$

$$t = \frac{v_0}{(v_0 + \omega_n x_0)\omega_n}$$



Under damped vibration

$0 < \zeta < 1$

$$s_1 = -\zeta\omega_n + i\sqrt{1-\zeta^2}\,\omega_n = -\zeta\omega_n + i\omega_d$$

$$s_2 = \zeta\omega_n - i\sqrt{1-\zeta^2}\,\omega_n = \zeta\omega_n - i\omega_d$$

where, $\omega_d = \sqrt{1-\zeta^2}\,\omega_n$

↑ freq. of damped free vibration or damped natural freq.

$$x(t) = e^{-\zeta\omega_n}\left(A_1 e^{i\omega_d} + A_2 e^{-i\omega_d}\right)$$

$$A_1 = A_2 = A_0 e^{-i\phi}$$

$$= A_0 e^{-\zeta\omega_n t}\left(e^{-i\phi} e^{i\omega_d t} + e^{i\phi}\right)$$

$$= A\, e^{-\zeta\omega_n t}\cos(\omega_d t - \phi)$$

$A = 2A_0$

Initial conditions are

$$x(t)\Big|_{t=0} = x_0, \quad \frac{dx(t)}{dt}\Big|_{t=0} = v_0$$

$$x_0 = A\cos\phi \quad \text{—— (1)}$$

$$\frac{dx}{dt} = A\left[e^{-\zeta\omega_n t}\cdot(-\zeta\omega_n)\cos(\omega_d t - \phi) - \sin(\omega_d t - \phi)\cdot\omega_d\, e^{-\zeta\omega_n t}\right]$$

$$v_0 = A\left[-\zeta\omega_n\cos\phi + \sin\phi\,\omega_d\right]$$

$$v_0 = A\left[\zeta\omega_n\frac{x_0}{A} + \sin\phi\,\omega_d\right]$$

$$v_0 = -\zeta\omega_n x_0 + A\sin\phi\,\omega_d$$

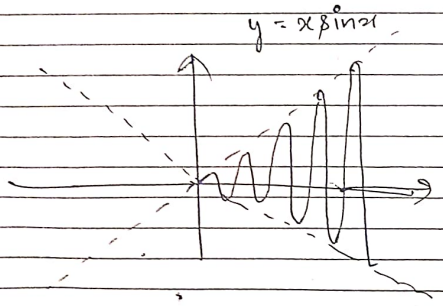
$$\frac{x_0}{A} = \cos\phi$$

$$\frac{v_0 + \zeta\omega_n x_0}{A\omega_d} = \sin\phi$$

$$\frac{\sin\phi}{\cos\phi} = \frac{\frac{V_0 - \zeta\omega_n x_0}{A\omega_d}}{\frac{x_0}{A}}$$

$$\boxed{\tan\phi = \frac{V_0 - \zeta\omega_n x_0}{x_0 \omega_d}}$$

$$\left(\frac{x_0}{A}\right)^2 + \left(\frac{V_0}{A\omega_d} - \frac{\zeta\omega_n x_0}{A\omega_d}\right)^2 = 1$$



$$A = \sqrt{x_0 + \frac{V_0}{\omega_d^2}}$$

$$\tan\phi = \frac{V_0}{\omega_d x_0}$$

$$x(t) = A e^{-\zeta\omega_n t} \cos(\omega_d t - \phi)$$

$$x(t)\Big|_{t=0} = x_0 \,, \quad \dot{x}(t)\Big|_{t=0} = V_0$$

$$x_0 = A\cos\phi$$

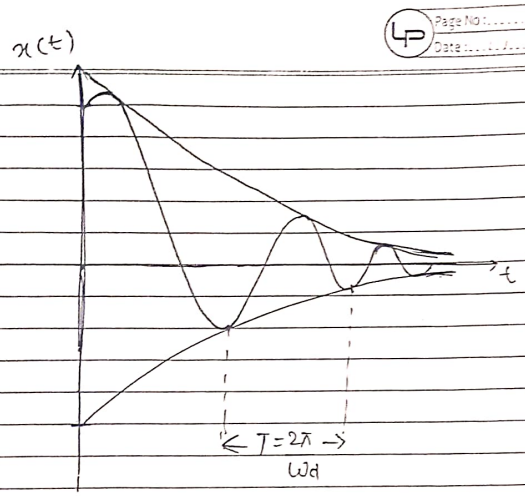
$$\dot{x}(t) = \left[A(-\zeta\omega_n)\cos(\omega_d t - \phi) - A\omega_d \sin(\omega_d t - \phi)\right]$$

Putting $t = 0$

$$A\sin\phi = \frac{V_0 + x_0 \zeta\omega_n}{\omega_d}$$

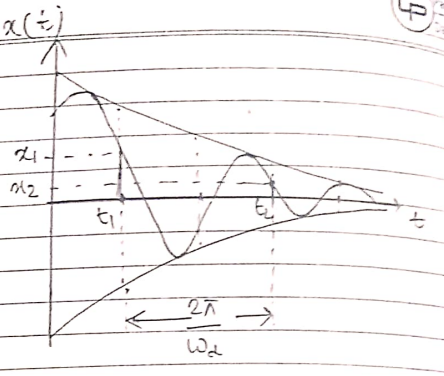
$$A = \sqrt{x_0^2 + \frac{(V_0 + x_0\zeta\omega_n)^2}{\omega_d^2}}$$

$$\tan\phi = \frac{V_0 + x_0\zeta\omega_n}{x_0\omega_d}$$



Logarithmic Decrement

Sometimes coefficient of Daming or Damping factor of a system is not known and needs to be find or determined experimentally.
In case of under damped vibration viscous damping causes the vibration decay exponentially, where the exponent is a linear function of b.



Let t_1 and t_2 are the times co... to two consecutive displacement... x_1 and x_2, measured one cycl... apart.

$$x_1 = A e^{-b\omega_n t_1} \cos(\omega_d t_1 - \phi)$$

$$x_2 = A e^{-b\omega_n t_2} \cos(\omega_d t_2 - \phi)$$

$$= A e^{-b\omega_n t_2} \cos(\omega_d t_1 + 2\pi - \dots$$

$$x_2 = A e^{-b\omega_n t_2} \cos(\omega_d t_1 - \phi)$$

$$\therefore t_2 - t_1 = \frac{2\pi}{\omega_d}$$

$$t_2 = \frac{2\pi}{\omega_d} + t_1$$

$$\frac{x_1}{x_2} = e^{\zeta \omega_n (t_2 - t_1)} = e^{\zeta \omega_n \frac{2\pi}{\omega_d}}$$

Say, $\delta = \ln\left(\frac{x_1}{x_2}\right) = \zeta \omega_n \frac{2\pi}{\omega_d}$

$$\delta = \frac{\zeta \omega_n 2\pi}{\sqrt{1-\zeta^2}\,\omega_n} \qquad \delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

$$\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}}$$

Q→ If t_1 and t_2 are 4 cycle apart write the expression of δ.

$$\zeta = \frac{\delta}{\sqrt{(8\pi)^2 + \delta^2}}$$

$$\delta = \ln\left(\frac{x_1}{x_2}\right) = \zeta \omega_n \frac{8\pi}{\omega_d}$$

Problem no. 1.34 (Pg-44) of book

Ans

$$\frac{8}{4} A_2 = \frac{8}{8} A_1 \left(\frac{3}{4}\right)$$

$$\frac{x_1}{x_2} = \frac{x_1}{0.75 x_1}$$

$$\delta = \ln \frac{x_1}{x_2} = \ln\left(\tfrac{4}{3}\right) = 0.287$$

$$\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} = 0.045$$

1.4, 1.6, (1.7) 1.10, 1.11, 1.13, 1.15, 1.18
1.20, 1.21, 1.22, 1.24, 1.25, 1.30,
17W

9/02/23

$m\ddot{x} + kx = 0$

$\omega_n = \sqrt{\frac{k}{m}}$

$x(t) = A\cos(\omega_n t - \phi)$

$m\ddot{x} + c\dot{x} + kx = 0$

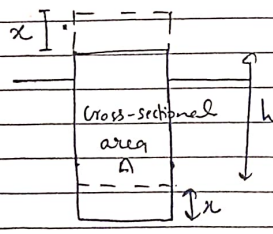
$\omega_n = \sqrt{\frac{k}{m}}$

$\zeta = \frac{c}{2m\omega_n}$

$\omega_d = \sqrt{1-\zeta^2}\,\omega_n$

$x(t) = A e^{-\zeta\omega_n t}\cos(\omega_d t - \phi)$

1.4



$m\ddot{x} = mg - \rho A g(h+x)$

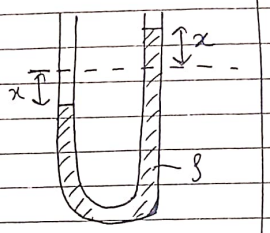
$m\ddot{x} + \rho A g x = 0$

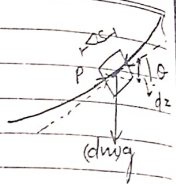
$m_{eq} = m$

$k_{eq} = \rho A g$

$\omega_n = \sqrt{\frac{\rho A g}{m}}$

1.6





$\frac{dm\,dv}{dt} = -\frac{\partial p}{\partial s} - dm g$

$\rho A \frac{dv}{dt}\Delta s = -A\frac{\partial p}{\partial s}\Delta s$

$\int \rho \frac{dv}{dt} ds = \int -\frac{\partial p}{\partial s} ds - \rho \int g$

∵ no external

$\rho \frac{dv}{dt} L = -\rho g 2x$

$L\ddot{x} + 2gx = 0$

$\ddot{x} + \frac{2g}{L}x = 0$

$k_{eq} = \frac{2g}{L}$

Energy Method

$K.E = \frac{1}{2} \rho V^2 (AL)$

$F = \rho g (2x) A$

Work done against gravity $= \int_0^x F \, dx$

$= \int_0^x 2 \rho g A x \, dx$

$= \rho g A x^2 = PE$

$U = K.E. + P.E.$

From the principle of Mechanical Energy Conservation,

$\Rightarrow \frac{dU}{dt} = 0$

$\frac{d}{dt}\left[\rho g (2xA) + \rho g A x^2\right] = 0$

$2 \rho g A \dot{x} + \rho g A (2) x \frac{dx}{dt} = 0$

$\dot{x} = \frac{dx}{dt} \qquad V = \frac{d^2x}{dt^2}$

$\frac{d}{dt}\left(\frac{1}{2} \rho V^2 (AL) + \rho g A x^2 \right)$

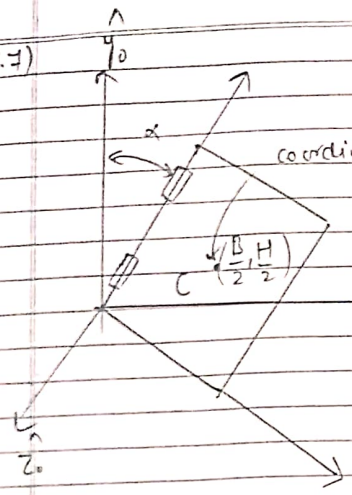
$\frac{1}{2} (\rho AL) 2V \frac{dV}{dt} + \rho g A \, 2x \frac{dx}{dt}$

$\rho AL \ddot{x} + \rho g A (2x) \dot{x} = 0$

$\ddot{x} + \frac{g}{L}$

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(1.7)



coordinates in from x_1, y_1, z_1

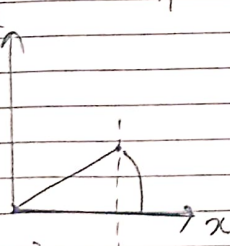
$\vec{r} \times$ momentum — angular momentum

Moment of momentum eqⁿ

or

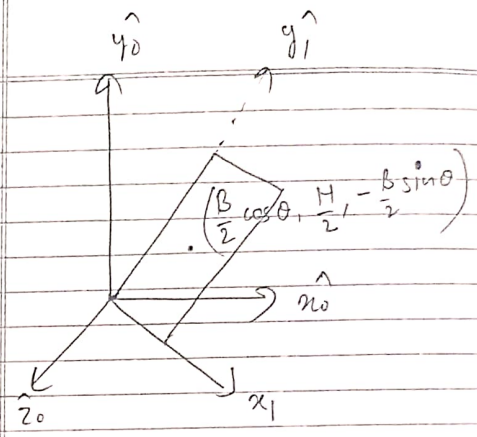
$\tau = I\alpha$ Euler-Newton eqⁿ of motion about y_1-axis.

either about com or Fixed axis



$$M\hat{y}_1 = (\vec{r_c} \times m\vec{g}) \cdot \hat{j}_1$$

$$\vec{r_c} \times mg(-\cos\alpha \hat{j}_1 + \sin\alpha \hat{i}_1) \cdot \hat{j}_1$$



$$M_{y_1} = \vec{r_c} \times mg(-\cos\alpha \hat{j}_1 + \sin$$

$$= \vec{r_c} \cdot mg(-\cos\alpha \hat{j}_1 + s$$

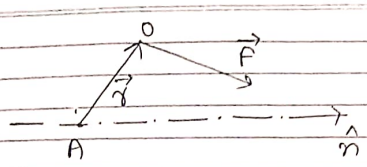
$$= \vec{r_c} \cdot mg\sin\alpha \hat{k}_1$$

$$= mg\left(\frac{B}{2}\cos\theta \hat{i}_1 + \frac{H}{2}\hat{j}_1 - \frac{B}{2}si\right.$$

$$= -mg\frac{B}{2}\sin\alpha\sin\theta$$

$$\boxed{\vec{a} \times \vec{b} \cdot \vec{c} = \vec{a} \cdot \vec{b} \times \vec{c}}$$

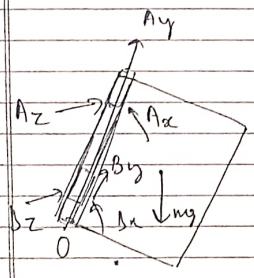
$$I\ddot{\theta} = mg\frac{B}{2}\sin\alpha\sin\theta$$



$\vec{M_A} = \vec{r} \times \vec{F}$

$M_n = \vec{M_A} \cdot \hat{n} = \vec{r} \times \vec{F} \cdot \hat{n}$

$= \vec{r} \cdot \vec{F} \times \hat{n} = \hat{n} \times \vec{r} \cdot \vec{F}$



① FBD

Eq^{n} of motion ② $F = ma$ COM $\perp^{r}$ (RB considered as Particle) (Force motion)

(Moment of Momentum Eq^{n}) ③ COM or about Euler's Newton Eq^{n}

ΣF_x

ΣF_y

Euler's Newton Eq^{n}

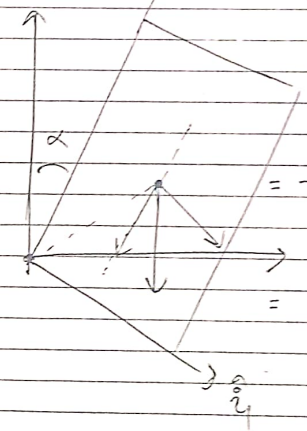
By Energy Method :→

$\frac{1}{2} m v_c^2 + \frac{1}{2} I \omega^2$ $\quad KE = \frac{1}{2} I_A \dot{\theta}^2$

∇ - Gradient of Potential Energy = Conservative force.

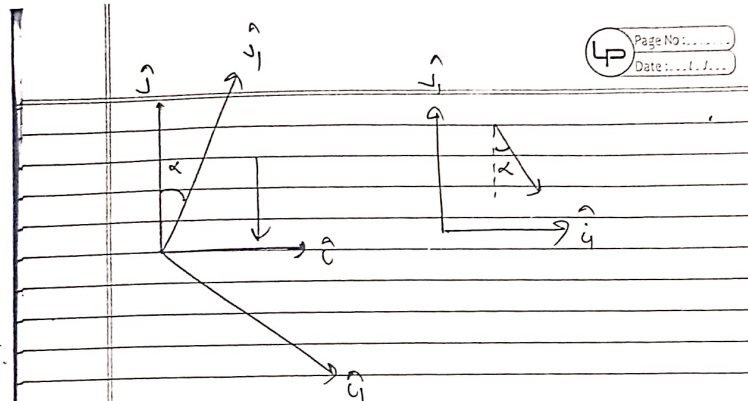
$-\frac{\partial V}{\partial r} = \vec{F}$

$V = -\int_{r_i}^{r_f} \vec{F} \cdot d\vec{r}$



$= -m\vec{g} \cdot \int_{r_i}^{r_f} d\vec{r}$

$= -m\vec{g} \cdot (\vec{r}_c(t) - \vec{r}_c(0))$



$$-mg\cos\alpha\,\hat{j}_1 + mg\sin\alpha\,\hat{i}_1$$

$$KE = \frac{1}{2} I_A \dot{\theta}^2$$

$$PE = -m\vec{g}\cdot\vec{r_c}$$

$$\left(g\sin\alpha\,\hat{i}_1 - g\cos\alpha\,\hat{j}_1\right)\cdot\left(\frac{B}{2}\cos\theta\,\hat{i}_1 + \frac{H}{2}\hat{j}_1\right)$$

$$-\left[mg\sin\alpha\,\frac{B\cos\theta}{2} - mg\cos\alpha\,\frac{H}{2}\right]$$

$$= mg\left[\frac{H\cos\alpha - B\sin\alpha\cos\theta}{2}\right]$$

$$U = KE + PE$$

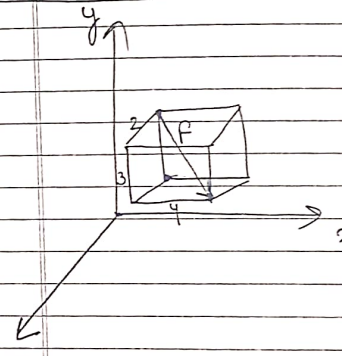
$$\frac{dU}{dt} = 0$$

$$\frac{d}{dt}(KE + PE) = 0$$

hw

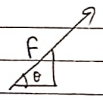
Ex-
10 eg

$$\frac{1}{2}$$



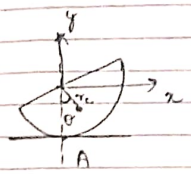
$$\vec{F} = F\,\frac{\vec{AB}}{|\vec{AB}|}$$

$$= F\left(\frac{2\hat{i} - 5\hat{j} -}{\sqrt{2^2 + 5^2 +}}\right)$$



$$F(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$$

Refrence point can be taken not changing with time.



$$PE = -mg\vec{r_c}$$

$$= -mg(-\hat{j}) \cdot \vec{r_c}$$

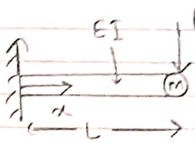
$$= mg\hat{j} \cdot \vec{r_c}$$

$$= mg\hat{j} \cdot \frac{2R}{\pi}(-\cos\theta\,\hat{j} + \sin\theta\,\hat{i})$$

$$= -\frac{2R}{\pi} mg\cos\theta$$

$$I_A\ddot{\theta} + mgr_c\sin\theta = 0$$

1.11



$$y = \frac{PL^3}{3EI}$$

Elastic potential energy developed in body,

$$\text{Work done} = \int_0^y P\,dy$$

$$= 3EI\int_0^y \frac{y}{L^3}\,dy$$

$$= \frac{1}{2}\left(\frac{3EI}{L^3}\right)y^2$$

$$= \frac{1}{2}k_{eq}y^2$$

$$K.E. = \frac{1}{2}m\dot{y}^2$$

$$U = \frac{1}{2}m\dot{y}^2 + \frac{1}{2}k_{eq}y^2$$

$$\frac{dU}{dt} = \frac{m}{2}2\dot{y}\ddot{y} + \frac{1}{2}k_{eq}2y\dot{y}$$

$$m\ddot{y} + k_{eq}y = 0$$

$$\omega_n = \sqrt{\frac{k_{eq}}{m}}$$

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once we know k_{eq} the eq^n

Inertia force + Elastic force = 0

$$m\ddot{y} + k_{eq}\, y = 0$$

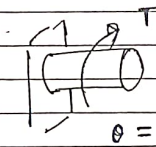
1.12



$$k_{eq} = \frac{GJ}{L}$$

$$\frac{T}{\theta} = k_{eq}$$

$$\int_0^\theta T\, d\theta$$



Flexure formula

$$\frac{T}{J} = \frac{G\theta}{L}$$

$$= \int_0^\theta \frac{G\theta J\, d\theta}{L}$$

$$= \frac{GJ}{L}\frac{\theta^2}{2}$$

$$= \frac{1}{2}\left(\frac{GJ}{L}\right)\theta^2$$

Stiffness = $\frac{\text{load}}{\text{Deformation}}$

$$= \frac{T}{\theta}$$

$$= \frac{GJ}{L} = k_{eq}$$

Eqn of motion = Inertia force + Elastic force

$$k_{eq} = \frac{GJ}{L}$$

$$KE = \frac{1}{2} I \dot{\theta}^2$$

$$\frac{d}{dt}\left[\frac{1}{2}\left(\frac{GJ}{L}\right)\theta^2 + \frac{1}{2} I \dot{\theta}^2\right] = 0$$

$$\frac{1}{2}\left(\frac{GJ}{L}\right) 2\theta\dot{\theta} + \frac{I}{2} 2\dot{\theta}\ddot{\theta} = 0$$

$$\ddot{\theta} + \frac{GJ}{LI}\theta = 0$$

$$\omega_n = \sqrt{\frac{GJ}{LI}}$$ Ans.

1.9 Answers

1.10 $h = 0.2\,m,\ 0.0717\,m$

1.13 $\omega_n = \sqrt{\frac{k_1}{m\left(1 + \frac{a^2}{b^2}\frac{k_1}{k_2}\right)}} = 30$ rad

1.14 $k_{eq} = \left(\frac{J_1}{L_1} + \frac{J_2}{L_2}\right)G$, $\omega_n =$

1.18 $\omega_n = \sqrt{\frac{1.15\,GJ}{IL}}$

1.19 $\omega_n = \sqrt{\frac{8EI}{3mL^3}}$

1.20) $\omega_n = \sqrt{\frac{2gb^2}{La^2}}$

Ans is wrong

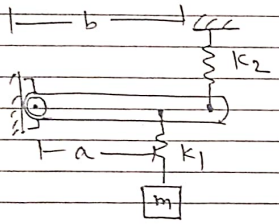
1.21) $\omega_n = \sqrt{\frac{KR^2}{mR^2+I}}$

1.28) $C_{eq} = CL^2$

$K_{eq} = mgL$ $\qquad \omega_n = \sqrt{\frac{K_{eq}}{m_{eq}}}$

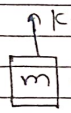
$m_{eq} = mL^2$

1.13



let deformation in spring ① $= x_1$

" " " ② $= x_2$



Eqⁿ of motion

$m\ddot{x} + K_1x_1 = 0$ —— (1)

$x = x_1 + x_A$ ← disp. of point A —— (2)

x_B = disp. of B $= x_2$

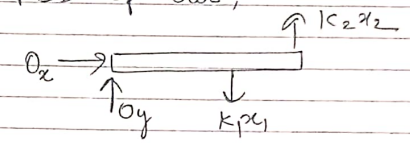
$b\theta = x_2$

$x_A = a\theta$

$x_A = \frac{a}{b}x_2$

$x = x_1 + \frac{a}{b}x_2$

FBD of bar,



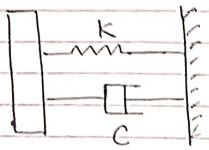
$\sum M_O = I\ddot{\theta} = 0$ (for mass less bar)

$K_2x_2\,b - K_1x_1\,a = 0$

$x_2 = \frac{a}{b}\frac{K_1}{K_2}x_1$

massless block

Q→



m = 10 kg
V = 50 m/s

If Damper in question, then never use energy conservation

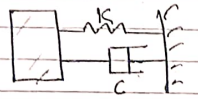
K = 64 × 10³ N/m
C = 400 Ns/m

Find max^m deflection in spring.

$mv \dot{x} = \frac{1}{2} K x^2 + c \dot{x}$

~~$KE = \frac{1}{2} m v^2$~~

~~$\frac{1}{2} K x^2$~~



mj

$\sqrt{x_0^2 + \frac{V_0^2}{\omega_d^2}}$ $\quad V_0 = 50$ m/s

$\omega_d = \sqrt{\quad}$

$A = \frac{V_0}{\omega_d}$

$\sqrt{(1 + f^2)\omega_n}$

$= \frac{50}{20\sqrt{15}}$

$\omega_n = \sqrt{\frac{K}{m}}$

$f = \frac{c}{2 m \omega_n} = \frac{c}{2\sqrt{m}\sqrt{K}}$

$f = \frac{c}{2\sqrt{mK}}$

$(f < 1)$

$\omega_d = 20\sqrt{15}$

$A = \boxed{0.65 \text{ m}}$

★ Eqⁿ of motion,

$m\ddot{x} + c\dot{x} + Kx = 0$

Solⁿ is given :-

$x(t) = A e^{-f \omega_n t} \cos(\omega_d t - \phi)$

where,

$\omega_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{64 \times 10^3}{10}} = 80$ rad/s

$f = \frac{c}{2 m \omega_n} = \frac{400}{2 \times 10 \times 80} = 0.25$

$\omega_d = \sqrt{1 - f^2}\, \omega_n = 77.46$

Initial conditions are,

$$x(t)\Big|_{t=0} = 0$$

$$0 = A\cos\phi$$

$$\phi = \pi/2$$

$$\dot{x}(t)\Big|_{t=0} = 50$$

$$\dot{x}(t) = A\left[-\zeta\omega_n \sin\omega_d t + \omega_d \cos\omega_d t\right] e^{-\zeta\omega_n t}$$

$$50 = A\omega_d$$

$$A = \frac{50}{\omega_d} = 0.65$$

For max^m deflection in spring

$$\dot{x}(t) = 0$$

$$\zeta\omega_n \sin\omega_d t = \omega_d \cos\omega_d t$$

$$\tan\omega_d t = \frac{\omega_d}{\zeta\omega_n}$$

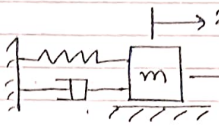
$$t = \frac{1}{\omega_d}\tan^{-1}\left(\frac{\omega_d}{\zeta\omega_n}\right) = 0.017 \text{ sec.}$$

$$x = Ae^{-\zeta\omega_n t}\sin\omega_d t$$

at $t = 0.017s$, $x_{max} = 0.444m$

'Geta' is just for solving problems.

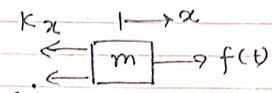
Response of SDOF system to Harmonic force



forcing freq

force amp.

FBD:



forces directing x will be -ve
forces increasing x will be +ve

$$\frac{c}{m} = 2\zeta\omega_n$$

$$m\ddot{x} = -Kx - c\dot{x} + f(t)$$

$$m\ddot{x} + c\dot{x} + Kx = f(t) = F_0\cos\Omega t$$

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{K}{m}x = \frac{F_0}{m}\cos\Omega t$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{f_0}{m}\cos\Omega t$$

$$x(t) = x_H(t) + x_p(t)$$

$x_H(t)$: Homogeneous sol^n or Transient response

$x_p(t)$: Particular Integral or (PI) steady state response

$x_H(t)$ is sol^n of eq^n

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$

$$x_H(t) = e^{-\zeta\omega_n t}\left(A\cos\omega_d t + B\sin\omega_d t\right) \quad \text{(Assuming } 0<\zeta<1\text{)}$$

For $x_p(t)$:

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = R_e\left(\frac{f_0}{m}e^{i\Omega t}\right)$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{f_0}{m}e^{i\Omega t} = \frac{\omega_n^2 f_0 e^{i\Omega t}}{K}$$

$$= \omega_n^2 x_{st} e^{i\Omega t}$$

x_{st} = static deformation of spring — ①

Assuming a solution, $x_p(t) = X e^{i\Omega t}$ — Putting this in eq^n ①

$$X\left[-\Omega^2 + i\Omega 2\zeta\omega_n + \omega_n^2\right]e^{i\Omega t} = \omega_n^2 x_{st} e^{i\Omega t}$$

$$X = \frac{\omega_n^2 x_{st} e^{i\Omega t}}{\omega_n^2 - \Omega^2 + 2i\zeta\Omega\omega_n} = \frac{x_{st} e^{i\Omega t}}{\left(1-\frac{\Omega^2}{\omega_n^2}\right) + 2i\zeta\frac{\Omega}{\omega_n}}$$

$$X = \frac{x_{st}\, e^{-i\phi}}{\left(\sqrt{\left(1-\frac{\Omega^2}{\omega_n^2}\right)^2 + 4\zeta^2\frac{\Omega^2}{\omega_n^2}}\right)}$$

$$\tan\phi = \frac{2\zeta\Omega/\omega_n}{\left(1-\frac{\Omega^2}{\omega_n^2}\right)}$$

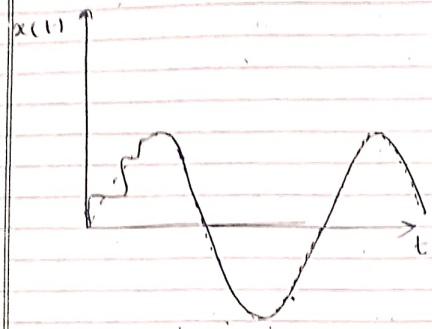
$$x_p(t) = \frac{x_{st}\, e^{-i\phi}\, e^{i\Omega t}}{\sqrt{\left(1-\frac{\Omega^2}{\omega_n^2}\right)^2 + 4\zeta^2\frac{\Omega^2}{\omega_n^2}}}$$

$$= \frac{x_{st}\, e^{i(\Omega t-\phi)}}{\sqrt{\left(1-\frac{\Omega^2}{\omega_n^2}\right)^2 + 4\zeta^2\frac{\Omega^2}{\omega_n^2}}}$$

$x(t) =$

After sometime transient response will die out, that time is known as :

$$T = \frac{1}{\zeta\omega_n} = \text{Time constant}$$



$$x_p(t) = \frac{x_{st}\, e^{i\Omega t}}{\left(1 - \frac{\Omega^2}{\omega_n^2}\right)^2 + i2\zeta\frac{\Omega}{\omega_n}} = G_1(i\Omega)\, x_{st}\, e^{i\Omega t}$$

FRF
(Frequency Response Factor)

$$G_1(i\Omega) = \frac{1}{1 - \frac{\Omega^2}{\omega_n^2} + i2\zeta\frac{\Omega}{\omega_n}} = \text{FRF}$$

$$\Rightarrow \text{FRF} = \frac{1}{\sqrt{\left(1 - \frac{\Omega^2}{\omega_n^2}\right)^2 + 4\zeta^2\frac{\Omega^2}{\omega_n^2}}}$$

$x + iy = \sqrt{x^2+y^2}\, e^{i\theta}$
$\tan\theta = y/x$

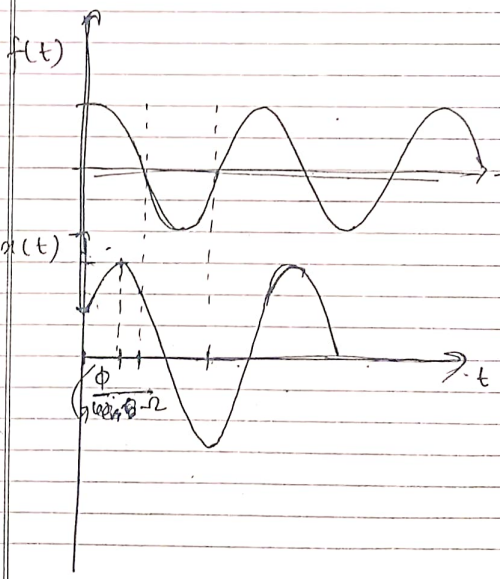
Magnitude of FRF is popularly known as Magnification factor.

$$= |G_1(i\Omega)|\, e^{-i\phi}$$

$$x_p(t) = |G(i\Omega)|\, x_{st}\, e^{i(\Omega t - \phi)}$$

$$= |G(i\Omega)|\, x_{st} \cos(\Omega t - \phi)$$

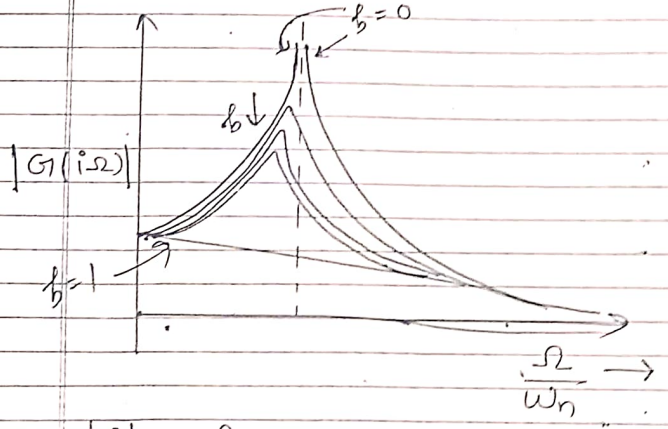
where, $\tan\phi = \frac{2\zeta\,\Omega/\omega_n}{\left(1 - \frac{\Omega^2}{\omega_n^2}\right)}$



We see:-

Phase difference and amplitude depends upon excitation frequency and amplitude of excitation.

Thus only two information necessary to characterize the harmonic response are :- amplitude and frequency of excitation, with time playing a secondary role.



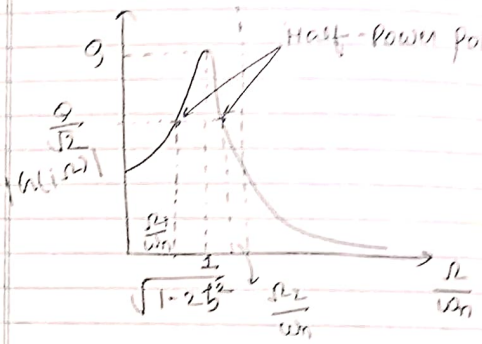
Let $\frac{\Omega}{\omega_n} = r$

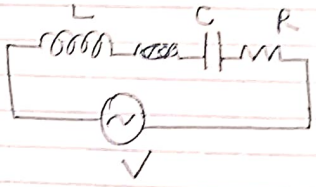
$$\frac{d}{dr}|G(ir)| = \frac{-4r(1-r^2) + 8\zeta^2 r}{\left[(1-r^2)^2 + 4\zeta^2 r^2\right]^{3/2}} = 0$$

$$r = \sqrt{1-2\zeta^2}$$

$$\frac{\Omega}{\omega_n} = \sqrt{1-2\zeta^2}$$

$$|G(i\Omega)|_{max} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$





$Q = CV$

$V(t) = L\frac{di}{dt}$

$= L\frac{d^2q}{dt^2}$

At half power point,

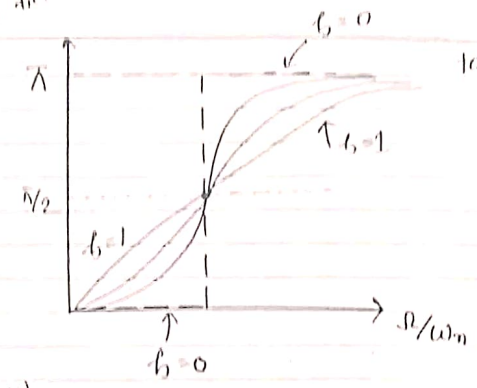
$$\frac{1}{\sqrt{\left(1-\frac{\Omega^2}{\omega_n^2}\right)^2+4\zeta^2\frac{\omega n^2}{\omega_n^2}}} = \frac{Q}{\sqrt{2}}$$

$$\Omega_2 - \Omega_1 = 2\zeta\omega_n$$

$Q = 5$

$\omega = 10$ rad/s

(i) ζ



(Fail safe design)

(Increasing difference Operation)

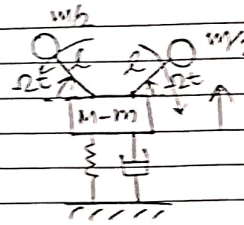
Velocity and force fluctuations freq. the same as that of natural freq. of the system.

Transmissibility $= \frac{2\zeta\Omega/\omega_n}{1-\frac{\Omega^2}{\omega_n^2}}$

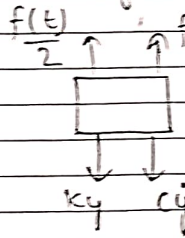
Q) 2.1, 2.2, 2.3, 2.4, 2.5

Rotating Imbalance

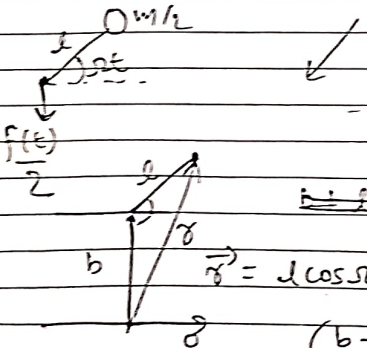
Unbalance → cause → Imbalance → result → Euler-Lagrange's Method



FBD of (M-m) mass,



$(M-m)\ddot{y} + c\dot{y} + ky = f(t)$ —— (1)



$\vec{r} = l\cos\Omega t\,\hat{i} + (b + y + l\sin\Omega t)\hat{j}$

$e^{i\theta} = \cos\theta + i\sin\theta$

$\frac{d\vec{r}}{dt} = -\Omega l\sin\Omega t\,\hat{i} + \left(\frac{dy}{dt} + \Omega l\right)$

$\frac{d^2\vec{r}}{dt^2} = -\Omega^2 l\cos\Omega t\,\hat{i} + \left(\frac{d^2y}{dt^2} - \right.$

$-\frac{m}{2}\Omega^2 l\cos\Omega t\,\hat{i} + \left(\frac{m\ddot{y}}{2} - \right.$

Writing eqn of motion in y-dir.

$\frac{m}{2}(\ddot{y} - l\Omega^2\sin\Omega t) = -\frac{f(t)}{2}$

$f(t) = m(\ddot{y} - l\Omega^2\sin\Omega t)$

Putting $f(t)$ in (1)

$(M-m)\ddot{y} + c\dot{y} + ky = m l\Omega^2$

$(M-m)\ddot{y}$

$M\ddot{y} + c\dot{y} + ky = m l\Omega^2 \sin\Omega t$

$\ddot{y} + 2\beta\omega_n\dot{y} + \omega_n^2 y = \frac{m l\Omega^2}{M}$

Assuming soln,

$y = A e^{i\Omega t}$

$y = A e^{i\Omega t}$

$\dot{y} = Ai\Omega e^{i\Omega t}$, $\ddot{y} = A(i\Omega)^2 e^{i\Omega t} = -A\Omega^2 e^{i\Omega t}$

$$A\left(-\Omega^2 + 2i\Omega\beta\omega_n + \omega_n^2\right) e^{i\Omega t} = \frac{m}{M} e \Omega^2 e^{i\Omega t}$$

$$A = \frac{\frac{m}{M} e \Omega^2}{\omega_n^2 \left[\left(1 - \frac{\Omega^2}{\omega_n^2}\right) + 2i\beta\frac{\Omega}{\omega_n}\right]}$$

$$A = \frac{m}{M} e \Omega^2 \, G(i\Omega)$$

$$A = \frac{m}{M} \frac{e \Omega^2}{\omega_n^2} \frac{e^{-i\phi}}{\sqrt{\left(1 - \frac{\Omega^2}{\omega_n^2}\right)^2 + 4\beta^2\frac{\Omega^2}{\omega_n^2}}}$$

$$\tan\phi = \frac{2\beta\Omega/\omega_n}{1 - \frac{\Omega^2}{\omega_n^2}}$$

$y = A e^{i\Omega t}$

$$\frac{m}{M} \frac{e \Omega^2}{\omega_n^2} \frac{e^{-i\phi} \times e^{i\Omega t}}{\sqrt{\left(1 - \frac{\Omega^2}{\omega_n^2}\right)^2 + 4\beta^2\frac{\Omega^2}{\omega_n^2}}}$$

$$y = \frac{m}{M} \frac{e \Omega^2}{\omega_n^2} |G(i\Omega)| \, e^{i(\Omega t - \phi)}$$

multiply & divide by $\frac{\Omega^2}{\omega_n^2}$

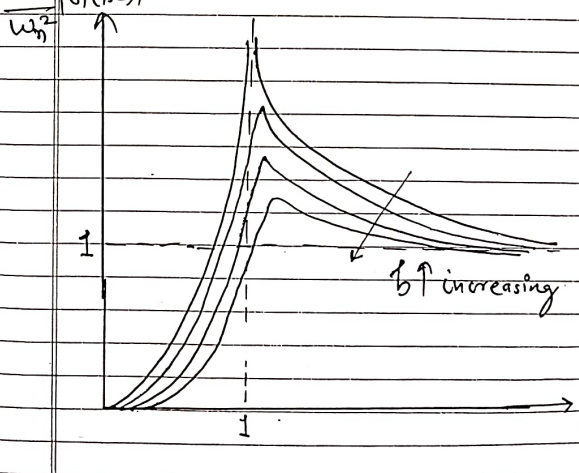
$$y = \frac{m}{M} e \frac{\Omega^2}{\omega_n^2} |G(i\Omega)| \sin(\Omega t - \phi)$$

$$y = \frac{m}{M} e \frac{\sin(\Omega t - \phi)}{\sqrt{\left(1 - \frac{\omega_n^2}{\Omega^2}\right)^2 + 4\beta^2\frac{\omega_n^2}{\Omega^2}}}$$

Amp. will be max^m at,

$$\frac{\Omega}{\omega_n} = \frac{1}{\sqrt{1 - 2\beta^2}}$$

$$(\text{Amp.})_{max} = \frac{1}{2\beta\sqrt{1-\beta^2}} \times \frac{me}{M}$$



Pg-103

2.12) $M = 12\,kg$

$\frac{m}{2} = 0.5\,kg \Rightarrow m = 1\,kg$

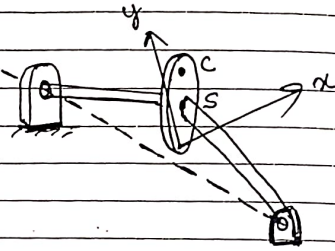
$0.1\ell = \frac{1}{2b\sqrt{1-b^2}} \frac{1}{12}\ell$

$2.4\,b\sqrt{1-b^2} = 1$

$b\sqrt{1-b^2} = \frac{5}{12}$

$\Omega = 300\ rpm$

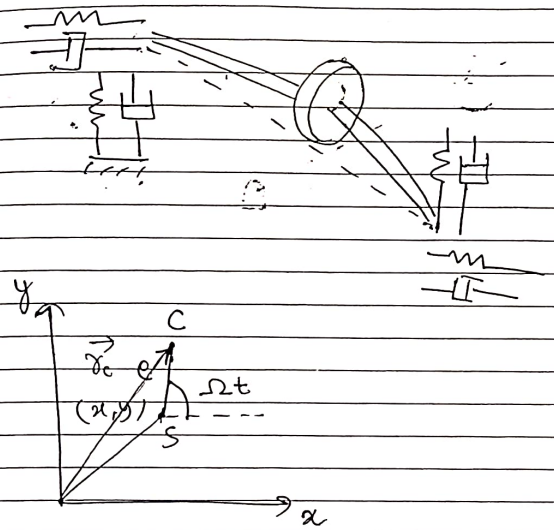
Whirling of rotating shaft



Many mechanical su

Many mechanical systems involve a rotating disk known as roto attached to a flexible shaft mount on bearing. Typical examples are ele motor, Turbine, etc.

If the rotor have some i.e. if the mass centre doesnot with the geometric centre then rotation produces a centrifugal fo causing the shaft to bend. Th of the plane containing the b shaft about the bearing axis known as whirling.



Rotation of shaft relative to inertial frame x-y = Ω

The mass of the disk = m

The eccentricity = e

Manufacturing fault मध्ये मध्ये eccentricity जास्त होते

$$\vec{r_c} = x\hat{i} + y\hat{j} + e\cos\Omega t\,\hat{i} + e\sin\Omega t\,\hat{j}$$

$$\vec{r_c} = (x + e\cos\Omega t)\hat{i} + (y + e\sin\Omega t)\hat{j}$$

Double Differentiate w.r.t time,

$$\ddot{\vec{r}}_c = (\ddot{x} - e\Omega^2\cos\Omega t)\hat{i} + (\ddot{y} - e\Omega^2\sin\Omega t)$$

Eqn of motion in x-direction,

Inertia force + damping force + Elastic force =

$$m(\ddot{x} - e\Omega^2\cos\Omega t) + c_x\dot{x} + k_x x = 0$$

$$m\ddot{x} + c_x\dot{x} + k_x x = me\Omega^2\cos\Omega t$$

$$\ddot{x} + 2\omega_{nx}b_x\dot{x} + \omega_{nx}^2 x = e\Omega^2\cos\Omega t$$

Similarly in y-direct^n eqn of motion is,

$$\ddot{y} + 2\omega_{ny}b_y\dot{y} + \omega_{ny}^2 y = e\Omega^2\sin\Omega t$$

$$x(t) = \frac{e\frac{\Omega^2}{\omega_{nx}^2}\cos(\Omega t - \phi_x)}{\sqrt{\left(1 - \frac{\Omega^2}{\omega_{nx}^2}\right)^2 + 4b_x^2\frac{\Omega^2}{\omega_{nx}^2}}}$$

where, $\tan\phi_x = \frac{2b_x\Omega/\omega_{nx}}{1 - \frac{\Omega^2}{\omega_{nx}^2}}$

$$y(t) = \frac{e\Omega^2}{\omega_{ny}^2} \sin(\Omega t - \phi_y)$$

$$\sqrt{\left(1 - \frac{\Omega^2}{\omega_{ny}^2}\right)^2 + 4\zeta_y^2 \frac{\Omega^2}{\omega_{ny}^2}}$$

$$\tan\phi_y = \frac{2\zeta_y \frac{\Omega}{\omega_{ny}}}{1 - \frac{\Omega^2}{\omega_{ny}^2}}$$

Amplitude in x-direct^n will be max^m when,

$$\frac{\Omega}{\omega_{nx}} = \frac{1}{\sqrt{1 - 2\zeta_x^2}}$$

$$(A_{mp.})_{max} = \frac{e}{2\zeta_x\sqrt{1 - \zeta_x^2}}$$

When,

$$C_x = C_y = 0$$

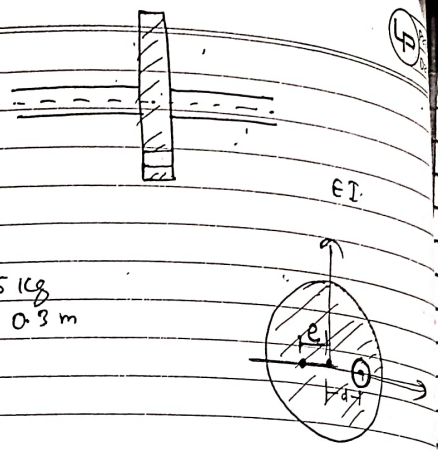
$$\tan\theta = \frac{y}{x} = \tan\Omega t$$

$$\theta = \Omega t$$

$$\dot{\theta} = \Omega$$

This proves that whole system will rotate like a rigid body.

2.13



$m = 15$ kg

$d = 0.3$ m

Fault simulator

Bearing

mass density =

$$\frac{A_1\bar{y}_1 - A_2\bar{y}_2}{A_1 - A_2}$$

$$\frac{\pi(0.3)^2(0) - \pi(0.03)^2(0.1}{\pi(0.3)^2 - \pi(0.03)^2}$$

$$\frac{-(0.03)^2(0.12)}{(0.3)^2 - (0.03)^2}$$

$$\frac{1.21}{1000}$$

$$= 0.00$$

$$- 0.001$$

$e = 0.00121\,m$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{48(1600)}{(0.4)^3(15)}} = 282.84\ rad/s$$

$\Omega = 6000\,rpm = 15.91\ rad/s$



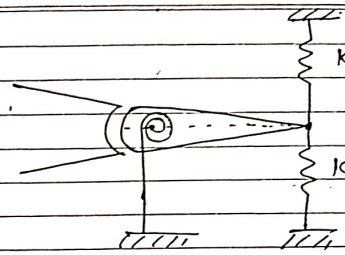
For simply supported beam: $\delta = \frac{PL^3}{48EI}$

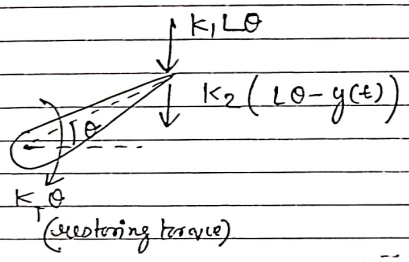
$$\frac{P}{\delta} = \frac{48EI}{L^3} = k_{eq}$$

$$\frac{e\frac{\Omega^2}{\omega_n^2}}{\sqrt{\left(1-\frac{\Omega^2}{\omega_n^2}\right)^2}}$$

$$\Rightarrow \frac{e\,\Omega^2/\omega_n^2}{1-\frac{\Omega^2}{\omega_n^2}} = 3.84\times10^{-6}\ m$$

Q.1





K_T

θ को घड़ी की दिशा में +ve

θ को घड़ी की विपरीत दिशा में -ve.

$$I_0\ddot{\theta} = -K_1L\theta L - K_2(L\theta - y(t))L - K_T\theta$$

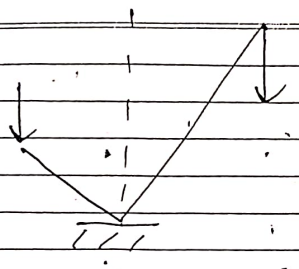
$$I_0\ddot{\theta} + K_T\theta + (K_1+K_2)L^2\theta = K_2L A\cos\omega t$$

$$\omega_r = \sqrt{\frac{(K_1+K_2)L^2 + K_T}{I_0}}$$

$$\omega_n = \sqrt{\frac{K_T}{I_0}}, \quad \omega_n^2 = \omega_r^2 - \frac{(K_1+K_2)L^2}{I_0}$$

Velocity and force maximum phase diff. $\pi/2$

(2.3)



(2.5)

2.4.5

Page No:.....
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Module-2

Variational Calculus.

$u = u(x)$

f

$\delta I(u) = \frac{\partial I}{\partial u} \delta u$

↓
variation

$\delta I(u,v) = \frac{\partial I}{\partial u} \delta u + \frac{\partial I}{\partial v} \delta v$

$u'(x) = \frac{du}{dx}$

$v'(x) = \frac{dv}{dx}$

$\delta I(u, v, u', v', x) = \frac{\partial I}{\partial u} \delta u + \frac{\partial I}{\partial u'} \delta u' + \frac{\partial I}{\partial v'}$

$\delta(\vec{u}.\vec{v}) = \delta\vec{u}.\vec{v} + \vec{u}.\delta\vec{v}$

$(\vec{F} - \vec{F_R}).\delta\vec{r} = 0$

(It may not be zero
virtual displacement
(Non-zero वैसे हो सकता है)

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Date:

d'Alembert's Principle

$$(m\ddot{\vec{r}} - \vec{F}) \cdot \delta\vec{r} = 0$$

$\vec{F} \cdot \delta\vec{r} = \delta W$

↓ (Virtual work)

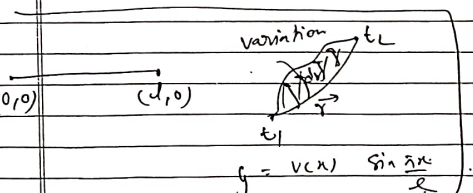
[virtual displacement]

$$\frac{d}{dt}(\dot{\vec{r}} \cdot \delta\vec{r}) = \ddot{\vec{r}} \cdot \delta\vec{r} + \dot{\vec{r}} \cdot \delta\dot{\vec{r}}$$

$$= \ddot{\vec{r}} \cdot \delta\vec{r} + \delta\left(\frac{1}{2}(\dot{\vec{r}} \cdot \dot{\vec{r}})\right)$$

$$m\frac{d\dot{\vec{r}}}{dt} \cdot \delta\vec{r} = m\ddot{\vec{r}} \cdot \delta\vec{r} + \delta\left(\frac{1}{2} m\dot{\vec{r}} \cdot \dot{\vec{r}}\right)$$

$$m\ddot{\vec{r}} \cdot \delta\vec{r} = -\delta T + m\frac{d\dot{\vec{r}}}{dt} \cdot \delta\vec{r}$$



$f = v(x) \sin\frac{\pi x}{l}$

Page No:
Date:

$$\left[-\delta T - \delta W + m\frac{d(\dot{\vec{r}} \cdot \delta\vec{r})}{dt}\right] \cdot \delta\vec{r} = 0$$

$$\int_{t_1}^{t_2} (\delta T + \delta W)\, dt = \int_{t_1}^{t_2} \left[m\frac{d}{dt}(\dot{\vec{r}} \cdot \delta\vec{r})\right] dt$$

$$\int_{t_1}^{t_2} \delta(T+W)\, dt = m(\dot{\vec{r}} \cdot \delta\vec{r})\Big|_{t_1}^{t_2}$$

$$\boxed{\int_{t_1}^{t_2} \delta(T+W)\, dt = 0}$$

Hamilton's Principle

[Conservative forces create -ve Potential.]

$$\delta \int_{t_1}^{t_2} (T - V)\, dt = 0$$

$\delta W = (\delta W)_C + (\delta W)_{NC}$ (Conservative, Non-conservative)

$(\delta W)_C = -\delta V$

$T - V = L$ ← Lagrangian

T = kinetic energy

V = Potential energy.

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11/08/23

Hamilton's Statement

The system follows that path

$$\mathcal{L} = L(x, \dot{x}, t)$$

for which

$$\delta \int_{t_1}^{t_2} \mathcal{L}(x, \dot{x}, t)\, dt = 0$$

Time is absolute

$$\delta \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial \dot{x}} \delta \dot{x} \right) dt = 0$$

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial x} \delta x\, dt + \underbrace{\frac{\partial L}{\partial \dot{x}} \delta x \Big|_{t_1}^{t_2}}_{0} - \int_{t_1}^{t_2} \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \delta x\, dt = 0$$

$$\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) \right) \delta x\, dt = 0$$

Involving the arbitrariness of δx.

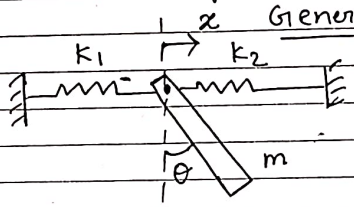
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0$$

↑ eqn of motion

[No. of Accuators = No. of DOF]
(Motor)

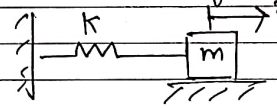
(Just count No. of accuators).

Generalised coordin



A set of minimum no. of coordina configura
Specify / describe a system Co
are known as generalised coo
the system.

No. of generalised cordinates =



$$T = \frac{1}{2} m \dot{x}^2$$

$$V = \frac{1}{2} k x^2$$

$$\mathcal{L} = T - V = \frac{1}{2} m \dot{x}^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}$$

$$\frac{\partial L}{\partial x} = -kx$$

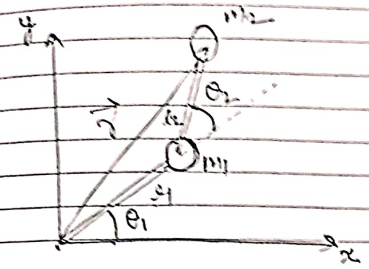
$$\frac{d}{dt}\left(\frac{\partial}{\partial}\right)$$

⇒ m

Q→ How to find generalized force?

Generalized coordinates

Generalized force
↓
w.r.t. if 1 force on system is change other co-ordinate will change etc.



$$\vec{r} = (l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2))\hat{i} + (l_1\sin\theta_1 + l_2\sin(\theta_1+\theta_2))\hat{j}$$

$$\vec{\delta r} = [-l_1\sin\theta_1\,\delta\theta_1 - l_2\sin(\theta_1+\theta_2)]\hat{i} + l_1\cos\theta_1\,\delta\theta_1 + l_2$$

$$\delta W = \vec{F}\cdot\vec{\delta r}$$

$$\delta W = F_x[\ \ ] + F_y[\ \ ]$$

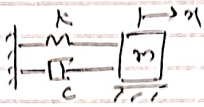
θ_1 ↑ θ_2

Generalised Force

$K = (c\dot{x})$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta} = Q_1$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} - \frac{\partial L}{\partial \theta} = Q_2$$



$\delta W = -c\dot{x}\,\delta x$

$m\ddot{x} + kx = -c\dot{x}$

$$\delta\vec{r} = [-l_1 s\theta_1\,\delta\theta_1 - l_2 s_{12}(\delta\theta_1+\delta\theta_2)]\hat{i} + [l_1 c\theta_1\,\delta\theta_1 + l_2 c_{12}(\delta\theta_1+\delta\theta_2)]\hat{j}$$

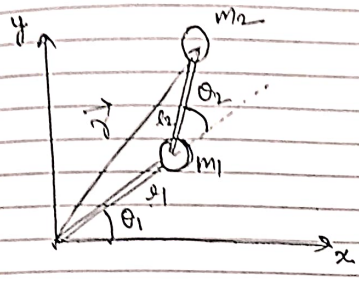
$$= (-F_x l s\theta_1 - F_x l_2 s_{12} + F_y l_1 c\theta_1 + F_y l_2 c_{12})\,\delta\theta_1 + (-F_x l_2 s_{12} + F_y l_2 c_{12})\,\delta\theta_2$$

↑ Q_1 ↑ Q_2

Generalised Force

Q. How to find generalized force?

Generalized coordinates



Generalized force
↓
वह सभी force जो system में change करने पर co-ordinate को भी change करे।

$$\vec{r} = (l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2))\hat{i} + (l_1\sin\theta_1 + l_2\sin(\theta_1+\theta_2))\hat{j}$$

$$\delta\vec{r} = [-l_1\sin\theta_1\,\delta\theta_1 - l_2\sin(\theta_1+\theta_2)]\hat{i}$$

$$+ l_1\cos\theta_1\,\delta\theta_1 + l_2 \cdots$$

$$\delta W = \vec{F}\cdot\delta\vec{r}$$

$$\delta W = F_x[\ \ ''\ \ ] + F_y[\ \ ''\ \ ]$$

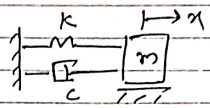
θ_1 ↑ ↑ θ_2

Generalised Force

$K = (c\dot{x})$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta} = Q_1$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} - \frac{\partial L}{\partial \theta} = Q_2$$



$$\delta W = -c\dot{x}\,\delta x$$

$$m\ddot{x} + kx = -c\dot{x}$$

$$\delta\vec{r} = [-l_1 s\theta_1\,\delta\theta_1 - l_2 s_{12}(\delta\theta_1+\delta\theta_2)]\hat{i} + [l_1 c\theta_1\,\delta\theta_1 + l_2 c_{12}(\delta\theta_1+\delta\theta_2)]\hat{j}$$

$$= (-f_x l s\theta_1 - f_x l_2 s_{12} + f_y l_1 c\theta_1 + f_y l_2 c_{12})\,\delta\theta_1 + (-f_x l_2 s_{12} + f_y l_2 c_{12})\,\delta\theta_2$$

$\delta\theta_1$ ↑ ↑ $\delta\theta_2$

Generalised Force

$$\int_{t_1}^{t_2} (\delta T - \delta V + \delta W_{nc})\, dt = 0$$

$$\delta W_{nc} = \vec{Q} \cdot \vec{\delta x}$$

$$\delta \int_{t_1}^{t_2} L\, dt + \int_{t_1}^{t_2} Q\, \delta x\, dt$$

$$\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \right) \delta x + \int_{t_1}^{t_2} Q\, \delta x\, dt = 0$$

Invoking the arbitrariness of δ.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = Q$$

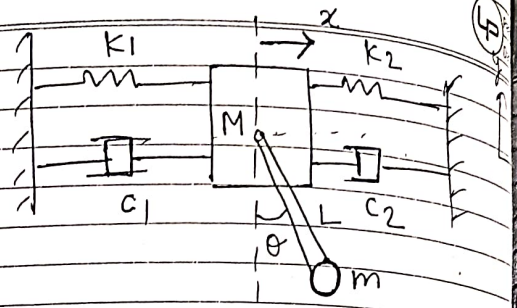
Rayleigh's dissipation factor

$$R = \frac{1}{2} c \dot{x}^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) + \frac{\partial R}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0$$

P-135

3.5



$$\frac{1}{2} M \dot{x}^2$$

$$V_x = -L\dot{\theta}\sin\theta$$

$$V_y =$$

$$-L\cos\theta\,\hat{i} \quad \& \quad -L\sin\theta\,\hat{j}$$

$$\vec{r_m} = (x + l\sin\theta)\,\hat{i} + (-l\cos\theta)\,\hat{j}$$

$$\vec{V_m} = \dot{\vec{r}}_m = (\dot{x} + l\dot{\theta}\cos\theta)\,\hat{i} + l\sin\theta\,\dot{\theta}\,\hat{j}$$

$$\vec{V_m} \cdot \vec{V_m} = \dot{x}^2 + l^2\dot{\theta}^2 + 2\dot{x}\dot{\theta} l\cos\theta$$

$$KE = T = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m \left(\dot{x}^2 + l^2\dot{\theta}^2 + \right.$$

$$PE = \frac{1}{2} k_1 x^2 + \frac{1}{2} k_2 x^2 - mgl\cos\theta$$

$$= \frac{1}{2} k x^2 - mgl\cos\theta \quad \text{[where}$$

Generalised co-ordinates

$$R = \frac{1}{2}c_1\dot{x}^2 + \frac{1}{2}c_2\dot{x}^2 = \frac{1}{2}c\dot{x}^2 \quad [c = c_1 + c_2]$$

$$L = T - V = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}ml^2\dot{\theta}^2 + m\dot{x}\dot{\theta}l\cos\theta = \frac{1}{2}kx^2 + mgl\cos\theta$$

Eqⁿ of motion :→

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \frac{\partial R}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} + \frac{\partial R}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

$\frac{\partial R}{\partial \dot{x}} = c\dot{x}\ddot{x}$, $\frac{\partial R}{\partial \dot{\theta}} = 0$

$\frac{\partial L}{\partial \dot{x}} = (M+m)\dot{x}\ddot{x}$ ⇒ $\frac{d}{dt}$

$\frac{\partial L}{\partial x} = -kx\dot{x}$

$\frac{d}{dt}(M+m)$

$\frac{d(x^n)}{dx} = nx^{n-1}$

uv

$u'v + v'u$

$\frac{\partial L}{\partial \dot{x}} = (M+m)\dot{x} + m\dot{\theta}l\cos\theta$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = (M+m)\ddot{x} + m\ddot{\theta}l\cos\theta - m\dot{\theta}^2 l\sin\theta$

$\frac{\partial L}{\partial x} = -kx$, $\frac{\partial R}{\partial \dot{x}} = c\dot{x}$

$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} + m\dot{x}l\cos\theta$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta} + m\ddot{x}l\cos\theta - m\dot{x}\dot{\theta}l\sin\theta$

$\frac{\partial L}{\partial \theta} = -mgl\sin\theta - m\ddot{x}\dot{\theta}l\sin\theta$

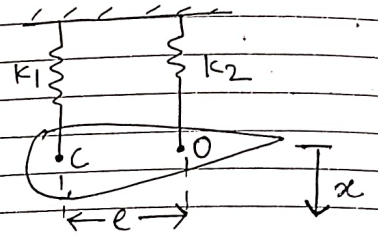
$\frac{\partial R}{\partial \dot{\theta}} = 0$

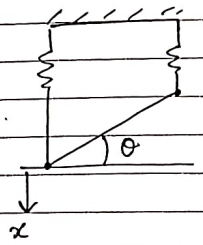
3.3, 3.4, 3.7, 3.11

3.5

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q-37





$\frac{1}{2} m v_c^2 +$

$ICE = \frac{1}{2} m v_c^2 + \frac{1}{2} I_c$

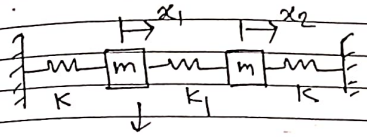
$= \frac{1}{2} m \dot{x}^2 + \frac{1}{2} I_c \dot{\theta}^2$

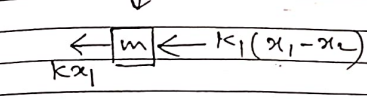
$PE = \frac{1}{2} K_1 x^2 + \frac{1}{2} K_2 (x - e\theta)$

Rayleigh's dissipation factor will be

13/09/23

Two-Degree-of-Freedom-Systems:-

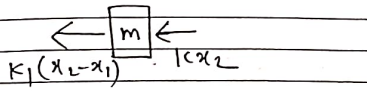




$$m\ddot{x}_1 + Kx_1 + K_1x_1 - K_1x_2 = 0$$

$$m\ddot{x}_1 + (K+K_1)x_1 - K_1x_2 = 0 \quad \text{---(1)}$$

Assuming $(x_1 > x_2)$



$$m\ddot{x}_2 + K_1(x_2 - x_1) + Kx_2 = 0$$

$$m\ddot{x}_2 - K_1x_1 + (K_1+K)x_2 = 0 \quad \text{---(2)}$$

Assuming $(x_2 > x_1)$

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} (K+K_1) & -K_1 \\ -K_1 & K+K_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0$$

mass matrix — Stiffness matrix

Assuming SOL^n :-

$$x_1 = A_1 e^{i\omega t}$$

$$x_2 = A_2 e^{i\omega t}$$

$$A\{u\} = \lambda I\{u\}$$

$$A - \lambda I = 0$$ ← eigen value form

$$\vec{x}(t) = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} e^{i\omega t}$$

Matrix is 0 when |Matrix| = 0

$$\ddot{\vec{x}}(t) = -\omega^2 \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} e^{i\omega t}$$

$$\begin{bmatrix} -m\omega^2 & 0 \\ 0 & -m\omega^2 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} + \begin{bmatrix} K+K_1 & -K_1 \\ -K_1 & K+K_1 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = 0$$

$$\begin{bmatrix} -m\omega^2 + K + K_1 & -K_1 \\ -K_1 & -m\omega^2 + K + K_1 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = 0 \quad \text{---(3)}$$

$$\left[(K+K_1) - m\omega^2\right]^2 - K_1^2 = 0$$

$$(K + K_1 - m\omega^2 - K_1)(K + K_1 - m\omega^2 + K_1) = 0$$

$$(K - m\omega^2)(2K_1 + K - m\omega^2) = 0$$

$$\omega_1 = \sqrt{K/m}$$

$$\omega_2 = \sqrt{\frac{K+2K_1}{m}}$$

Put $\omega = \omega_1$ in eqn ③,

$$\begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = 0$$

$$k_1 A_1 - k_1 A_2 = 0$$

$$A_1 = A_2$$

Eigen vector corresponding ω_1

$$= \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \quad \text{Put } A_1 = 1.$$

which is mode I

Put $\omega = \omega_2$

$$\begin{bmatrix} -m\omega_2^2 + k + k_1 & -k_1 \\ -k_1 & -m\omega_2^2 + k + k_1 \end{bmatrix}$$

$$\omega_2^2 = \frac{k + 2k_1}{m}$$

$$\begin{bmatrix} -(k+2k_1) + k + k_1 & -k_1 \\ -k_1 & -(k+2k_1) + k + k_1 \end{bmatrix}$$

$$\begin{bmatrix} -k_1 & -k_1 \\ -k_1 & -k_1 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = 0$$

Eigen vector corresponding to ω_2 $= \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$

$-k_1 A_1 - k_1 A_1 = 0$

$A_1 = -A_2$

Put $A_1 = 1$,

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \left(a_1 e^{i\omega_1 t} + \bar{a}_1 e^{-i\omega_1 t} \right) \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} +$$

$$\left(a_2 e^{i\omega_2 t} + \bar{a}_2 e^{-i\omega_2 t} \right) \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

$$= \left(c_1 \cos \omega_1 t + s_1 \sin \omega_1 t \right) \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} +$$

$$= \left(c_2 \cos \omega_2 t + s_2 \sin \omega_2 t \right) \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

at $t = 0$,

$x_1(0) = a \qquad \dot{x}_1(0) = 0$

$x_2(0) = a \qquad \dot{x}_2(0) = 0$

$$\begin{Bmatrix} a \\ a \end{Bmatrix} = c_1 \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + c_2 \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

$c_1 + c_2 = a$

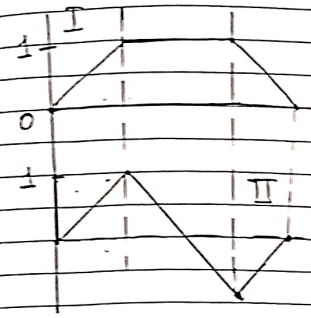
$c_1 - c_2 = a$

$c_1 = a$

eigen value - frequency

eigen vector - mode

Mode shape



Modal space α_2
or
Vector space

$\omega_1 = \sqrt{\frac{k}{m}} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

x_1

$\omega_2 = \sqrt{\frac{k+2k_1}{m}}$

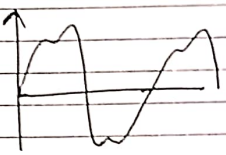
$\begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$

If we give arbitrary displacement then it is very to predict the motion but $\sqrt{\frac{k}{m}} = \omega_1$ and $\sqrt{\frac{k+2k_1}{m}} = \omega_2$

$\therefore C_1 e^{i\omega_1 t} + (C_1 + C_2 t) e^{i\omega_2 t}$

Ref Ed. use

are commensurate and the modal space will give a closed curve.



Example-3.1,
(Pg-115 book)

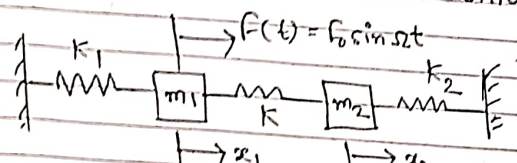
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Initial ...

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Example

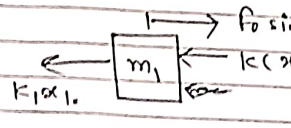
Two DOF system with Harmonic Excitation



If it is subjected to

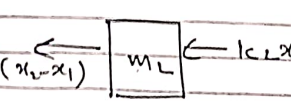
→ One is Harmonic excitation.
→ One is initial excitation

In this we are giving harmonic excitation.



$$m_1\ddot{x}_1 + k_1x_1 + k(x_1 - x_2) = F_0 \sin\Omega t$$

$$\boxed{m_1\ddot{x}_1 + (k_1 + k)x_1 - kx_2 = F_0 \sin\Omega t} \quad \text{—— (1)}$$



$$m_2\ddot{x}_2 + k(x_2 - x_1) + k_2x_2 = 0$$

$$\boxed{m_2\ddot{x}_2 - kx_1 + (k + k_2)x_2 = 0} \quad \text{—— (2)}$$

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$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k & -k \\ -k & k + k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

Assuming Sol^n :-

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} \sin\Omega t$$

$$\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} = -\Omega^2 \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} \sin\Omega t$$

$$\begin{bmatrix} -m_1\Omega^2 & 0 \\ 0 & -m_2\Omega^2 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k & -k \\ -k & k_2 + k \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} =$$

$$\begin{bmatrix} -m_1\Omega^2 + k_1 + k & -k \\ -k & k_2 + k - m_2\Omega^2 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} =$$

finding the inverse of the matrix,

$$\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \frac{1}{\left(-m_1\Omega^2 + k_1 + k\right)\left(-m_2\Omega^2 + k + k_2\right) - k^2}$$

$$\begin{bmatrix} k_2 + k - m_2 \\ k \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj}(A)}{|A|} \qquad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\text{adj}(A) = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

$$A_1 = \frac{f_0(-m_2\Omega^2 + k_2 + k)}{(-m_1\Omega^2 + k_1 + k)(-m_2\Omega^2 + k_2 + k) - k^2}$$

$$A_2 = \frac{kf_0}{(-m_1\Omega^2 + k_1 + k)(-m_2\Omega^2 + k_2 + k) - k^2}$$

we have dropped the homogeneous solⁿ part.

⇒ particular 'x' के A_1 'o' आएगा।

$$A_1 = \frac{f_0(-m_2\Omega^2 + k_2)}{(-m_1\Omega^2 + k_1 + k_2)(-m_2\Omega^2 + k_2) - k_2^2}$$

A_2 =

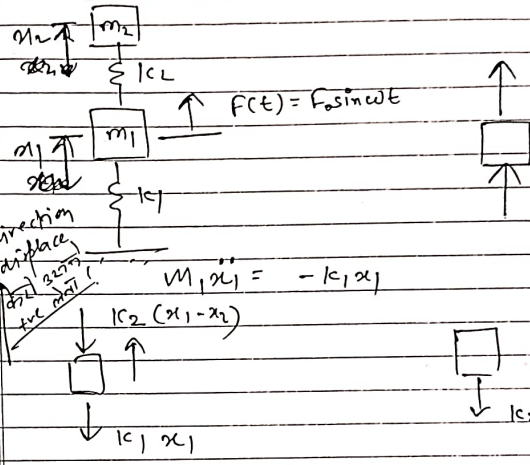
Elastic coupling
Elastic matrix
Non diagonal.

$k_2 = m_2\omega^2$

$\Rightarrow \omega = \sqrt{\frac{k_2}{m_2}}$

↳ vibration absorb करने के लिए।

Undamped Vibration Absorber



x_1 direction displace हो तो spring में force

$$m_1\ddot{x}_1 = -k_1 x_1$$

$$m_1\ddot{x}_1 + k_1 x_1 + k_2(x_1 - x_2) = F_0 \sin\omega t$$

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = F_0 \sin\omega t \quad \text{——①}$$

$$m_2\ddot{x}_2 + k_2(x_2 - x_1) = 0$$

$$m_2\ddot{x}_2 - k_2 x_1 + k_2 x_2 = 0 \quad \text{——②}$$

Elastic coupling
Elastic matrix

mass matrix

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -m_1\Omega^2 + k_1 + k_2 & -k_2 \\ -k_2 & -m_2\Omega^2 + k_2 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} F_0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \frac{1}{(-m_1\Omega^2 + k_1 + k_2)(-m_2\Omega^2 + k_2) - k_2^2} \begin{bmatrix} -m_2\Omega^2 + k_2 & k_2 \\ k_2 & -m_1\Omega^2 + k_1 \end{bmatrix} \begin{Bmatrix} F_0 \\ 0 \end{Bmatrix}$$

$$A_1 = \frac{f_0(-m_2\Omega^2 + k_2 + k)}{(-m_1\Omega^2 + k_1 + k)(-m_2\Omega^2 + k_2 + k) - k^2}$$

$$A_1 = \frac{k f_0}{(-m_1\Omega^2 + k_1 + k)(-m_2\Omega^2 + k_2 + k) - k^2}$$

we have dropped the homogeneous soln part.

⇒ particular 'x' में A_1 'o' आएगा।

$$A_1 = \frac{f_0(-m_2\Omega^2 + k_2)}{(-m_1\Omega^2 + k_1 + k_2)(-m_2\Omega^2 + k_2) - k_2^2}$$

$A_2 =$

Elastic coupling

Elastic matrix ↳ for diagonal.

$k = m\omega^2$

$\Rightarrow \omega = \sqrt{\frac{k_2}{m}}$

↳ Vibration absorb करने के लिए!

Undamped Vibration Absorber

m_2 m_1 k_2 k_1 $F(t) = F_0 \sin\omega t$

$k_2(x_1 - x_2)$

$k_1 x_1$

$m_1\ddot{x}_1 = -k_1 x_1$

$k_2(x_1 - x_2)$

$k_2(x_2 - x_1)$

$k_1 x_1$

$$m_1\ddot{x}_1 + k_1 x_1 + k_2(x_1 - x_2) = F_0 \sin\omega t$$

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = F_0 \sin\omega t \quad \text{——(1)}$$

$$m_2\ddot{x}_2 + k_2(x_2 - x_1) = 0$$

$$m_2\ddot{x}_2 - k_2 x_1 + k_2 x_2 = 0 \quad \text{——(2)}$$

Elastic matrix

mass matrix

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -m_1\Omega^2 + k_1 + k_2 & -k_2 \\ -k_2 & -m_2\Omega^2 + k_2 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} F_0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \frac{1}{(-m_1\Omega^2 + k_1 + k_2)(-m_2\Omega^2 + k_2) - k_2^2} \begin{bmatrix} -m_2\Omega^2 + k_2 & k_2 \\ k_2 & -m_1\Omega^2 + k_1 + k_2 \end{bmatrix} \begin{Bmatrix} F_0 \\ 0 \end{Bmatrix}$$

Beside the solution of derivation of 2 DOF Harmonic excitation.

In this solution, the homogenous part has not been written which doesn't mean that it has been ignored. However, normally real situations comes with dampers. One may say that for the sake of simplicity the damper has been dropped.

For. $\Omega = \sqrt{\frac{k_2}{m_2}}$

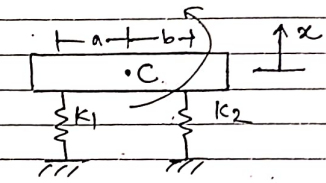
$A_2 = \frac{-f_0}{k_2}$

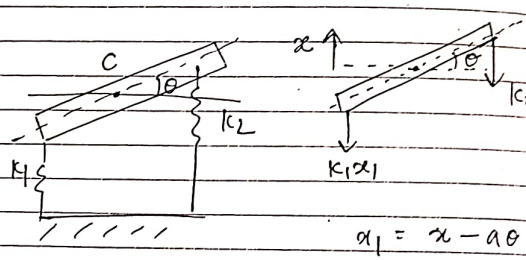
Note for the 2nd question of 2DOF undamped vibration:-

In the solution for the vibration to be completely absorbed, operating frequency of primary system should be $\sqrt{\frac{k_2}{m_2}}$, in that case static deformation will be $\frac{-f_0}{k_2}$

Dynamics of Deformable body

Coordinate coupling





$x_1 = x - a\theta$

$x_2 = x + b\theta$

$m\ddot{x} + k_1x_1 + k_2x_2 = 0$

$m\ddot{x} + k_1(x - a\theta) + k_2(x + b\theta) = 0$

$m\ddot{x} + (k_1 + k_2)x + (-k_1a + k_2b)\theta = 0$

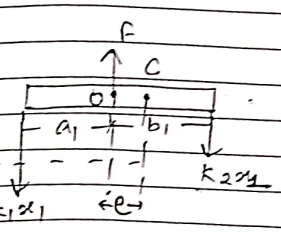
Writing moment of momentum eq^n about C is the centre of mass of

$I_c\ddot{\theta} = k_1x_1a - k_2x_2b$

$I_c\ddot{\theta} + (-k_1a + k_2b)x + (k_1a^2 + k_2b^2)\theta =$

$$\begin{bmatrix} m & 0 \\ 0 & I_c \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_1a + k \\ -k_1a + k_2b & k_1a^2 + k \end{bmatrix}$$

Static Equilibrium



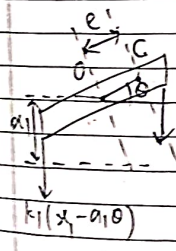
$\Sigma M_O = 0$

$k_1x_1a_1 = k_2x_1b_1$

$k_1a_1 = k_2b_1$

$k_1(a-e) = k_2(b+e)$

$$(k_1+k_2)e = -(-k_1a + k_2b)$$



(Transformation matrix) $x = x_1 + e\theta$

$\theta = \theta$

$$\begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{bmatrix} 1 & e \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ \theta \end{Bmatrix}$$

$$M\ddot{X} + KX = 0$$

(stiffness matrix always symmetric)

M = Mass matrix

K = Stiffness matrix

X = Displacement vector.

9/10/23

$$X = \begin{Bmatrix} x \\ \theta \end{Bmatrix}, \quad X_1 = \begin{Bmatrix} x_1 \\ \theta \end{Bmatrix}$$

$$X = \begin{bmatrix} 1 & e \\ 0 & 1 \end{bmatrix} X_1$$

Differentiate (DES) twice w.r.t time,

$$\ddot{X} = \begin{bmatrix} 1 & e \\ 0 & 1 \end{bmatrix} \ddot{X}_1 = P\ddot{X}_1$$

$$M\ddot{X} + KX = 0 \quad \text{---} (1)$$

$$MP\ddot{X}_1 + KPX_1 = 0 \quad \text{---} (2)$$

$$MP = \begin{bmatrix} m & 0 \\ 0 & I_c \end{bmatrix} \begin{bmatrix} 1 & e \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} m & me \\ 0 & I_o \end{bmatrix}$$

Now,

$$P^T MP = \begin{bmatrix} 1 & 0 \\ e & 1 \end{bmatrix} \begin{bmatrix} m & me \\ 0 & I_c \end{bmatrix}$$

$$= \begin{bmatrix} m & m.e \\ me & I_c + e^2m \end{bmatrix}$$

$$= \begin{bmatrix} m & me \\ me & I_o \end{bmatrix}$$

$$KP = \begin{bmatrix} k_1+k_2 & -k_1a+k_2b \\ -k_1a+k_2b & k_1a^2+k_2b^2 \end{bmatrix} \begin{bmatrix} 1 & e \\ 0 & 1 \end{bmatrix}$$

[Physical Quantities vectors, Law of addition & Polygon law]

$$= \begin{bmatrix} k_1+k_2 & (k_1+k_2)e - k_1a + k_2b \\ -k_1a+k_2b & (-k_1a+k_2b)e + k_1a^2 + k_2b^2 \end{bmatrix}$$

$$= \begin{bmatrix} k_1+k_2 & 0 \\ -k_1a+k_2b & k_1a_1^2 + k_2b_1^2 \end{bmatrix}$$

$$P^T K P = \begin{bmatrix} 1 & 0 \\ e & 1 \end{bmatrix} \begin{bmatrix} k_1+k_2 & 0 \\ -k_1a+k_2b & k_1a_1^2 + k_2b_1^2 \end{bmatrix}$$

$$P^T K P = \begin{bmatrix} k_1+k_2 & 0 \\ 0 & k_1a_1^2 + k_2b_1^2 \end{bmatrix}$$

Pre-multiplying eq^n ② with P^T,

$$P^T M P \ddot{X}_1 + P^T K P X_1 = 0$$

$$\begin{bmatrix} m & me \\ me & I_0 \end{bmatrix} \begin{Bmatrix} \ddot{\alpha}_1 \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & 0 \\ 0 & k_1a_1^2 + k_2b_1^2 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \theta \end{Bmatrix} = 0$$

mass matrix — Inertia / Dynamic Coupling

Inertia coupling
or
dynamic coupling

System can have → from ... elastic coupling is possible

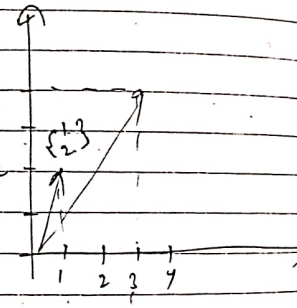
This exercise help us recognize namely coupling is not an inher characteristic of a system but of the coordinates used to des the motion of the system.

Mass matrix - diagonal -
Stiffness matrix - diagonal -

The co-ordinates corresponding to the equations of motion are known as co-ordinates or principle co-ordinates. The natural co-ordinates are unique System.

$$\vec{a} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$$

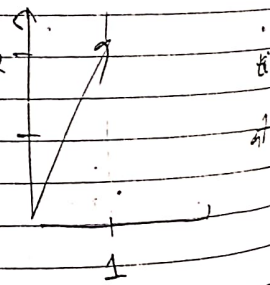
$$\vec{b} = \begin{Bmatrix} 3 \\ 4 \end{Bmatrix}$$



$$\vec{b} = P\vec{a}$$

$$\vec{b} = P\vec{a} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$$

New vector space.



Shear flow

$$\begin{bmatrix} \gamma \\ 0 \end{bmatrix}$$

Shear flow

Amount of Shearing is

Prerequisites

$a = \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$, $b = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix}$

Inner product,

$\vec{a} \cdot \vec{b} = a^T b = a_1 b_1 + a_2 b_2 + a_3 b_3$

$(AB)^T = B^T A^T$

$(ABC)^T = C^T B^T A^T$

Eigen vector of symmetric matrix are orthogonal.

→ Let A is a symmetric matrix.

Eigen pairs = $\{(\lambda_1, u_1), (\lambda_2, u_2) \dots (\lambda_i, u_i)\}$

$Au_i = \lambda_i u_i$

Multiply by u_j^T:

$u_j^T A u_i = u_j^T \lambda_i u_i \iff u_i^T A u_j = u_i^T \lambda_j u_j$ — (1)

$= \lambda_i u_j^T u_i$

(New vector share dot represent karta hai)

Taking Transpose both sides :-

$u_i^T A u_j = \lambda_j u_i^T u_j$

$\lambda_j u_j^T u_i$

(Dot products are commutative)

$u_i^T \lambda_j u_j$

$\Rightarrow (\lambda_j - \lambda_i) u_j^T u_i = 0$

Important for exam.

This proves if $i \neq j$, then u_i and u_j are orthogonal.

If $i \neq j$ ~~u_i and u_j~~ as well as $(\lambda_i = \lambda_j)$ then u_i and u_j are orthogonal.

Orthogonality of modal vectors

$M\ddot{X} + KX = 0$

$-M\omega^2 X + KX = 0$ (eigen value corresponds eigen pairs)

Eigen pairs :- $\{(\omega_1^2, u_1), (\omega_2^2, u_2), \dots (\omega_i^2, u_i)\}$

$u_j^T K u_j = -\omega_j^2 u$

we are choosing eigen vectors such that

$u_i^T M u_i = 1$

$-M\omega_i^2 u_i + K u_i = 0$

$-u_j^T (M\omega_i^2) u_i + u_j^T K u_i = 0$ — (1)

$-u_i^T (M\omega_i^2) u_j + u_i^T K u_j = 0$ — (2) Take transpose both side.

~~$-u_i^T (M\omega_j^2) u_j + u_i^T K u_j = 0$~~ — (3)

$$\begin{Bmatrix}1\\1\end{Bmatrix}, \begin{Bmatrix}1\\-1\end{Bmatrix}$$

$$A^{I} = \begin{Bmatrix}\alpha\\ \alpha\end{Bmatrix}$$

$$\boxed{(A^{I})^{T} M A^{I} = 1}$$

$$\begin{bmatrix}\alpha & \alpha\end{bmatrix}\begin{bmatrix}m & 0\\ 0 & m\end{bmatrix}\begin{Bmatrix}\alpha\\ \alpha\end{Bmatrix} = 1$$

$$\alpha = \frac{1}{\sqrt{2m}}$$

$$A^{I} = \frac{1}{\sqrt{2m}}\begin{Bmatrix}1\\1\end{Bmatrix}$$

$$A^{II} = \frac{1}{\sqrt{2m}}\begin{Bmatrix}1\\-1\end{Bmatrix}$$

$$u_j^T K u_j = 1$$

$$-M\omega_j^2 u_j + K u_j = 0$$

$$-u_i^T(M\omega_j^2)u_j + u_i^T K u_j = 0$$

$$-u_j^T(M\omega_j^2)u_j + u_j^T K u_i = 0$$

$$-u_i^T(M\omega_j^2)u_j + u_i^T K u_j = 0$$

$$(u_i^T - u_j^T)(M\omega_j^2)(u_i - u_j)$$

$$-u_j^T(M\omega_i^2)u_i + u_j^T K u_i = 0$$

$$-u_i^T(M\omega_j^2)u_j + u_i^T K u_j = 0$$

&

$$-u_2^T(M\omega_1^2)u_1 + u_2^T K u_1 = 0$$

$$-u_1^T(M\omega_2^2)u_2 + u_1^T K u_2 = 0$$

writing i as 1 and j as 2

$u_2^T K u_2 = 1$ $\quad u_1^T$

$$-u_2^T (M\omega_1^2) u_1 + u_2^T K u_1 = 0 \quad \text{——①}$$

Take Transpose

$$-u_1^T (M\omega_1^2) u_2 + u_1^T K u_2 = 0 \quad \text{——②}$$

replace 1 and 2

$$-u_1^T (M\omega_2^2) u_2 + u_1^T K u_2 = 0 \quad \text{——③}$$

subtract ② − ③

$$-u_1^T M(\omega_1^2 - \omega_2^2) u_2 = 0$$

Modal vectors are orthogonal w.r.t mass matrix.

u_j remains on same plane even after multiplying with mass matrix.

This proves:

If $j \neq i$ (as well as $\omega_i \neq \omega_j$)

$$u_i^T M u_j = 0$$

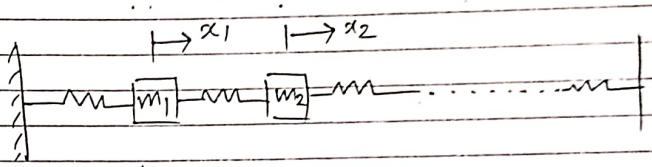
This proves Modal vectors are orthogonal w.r.t mass matrix.

Putting this result: $u_i^T M u_j = 0$ ——④ in eqn ③

we see,

$$u_i^T K u_j = 0$$

This proves Modal vectors are orthogonal w.r.t stiffness matrix too.



$$M\ddot{X} + KX = 0$$

$$M = \begin{bmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m_n \end{bmatrix}, \quad K = \begin{bmatrix} k_1+k_2 & -k_2 & \cdots & \\ -k_2 & \cdots & & \\ \cdots & & & \end{bmatrix}$$

$$X = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}$$

we determine eigen values and eigen vectors.

Eigen pairs $= \{(\omega_1^2, u_1), (\omega_2^2, u_2) \cdots\}$

$u_1 \rightarrow$ model vector of $n \times 1$

$u_n \rightarrow$ model vector of $n \times 1$

$P = [u_1 \; u_2 \; \dots \; u_n]$ ($n \times n$ dimension)

$u_i^T M u_i = 1$

$u_i^T K u_i = w_i^2$

$X = PY$

$M P \ddot{Y} + K P Y = 0$

Pregmultiply with P^T

$P^T M P \ddot{Y} + P^T K P Y = 0$

$$\begin{bmatrix} 1 & 0 & \dots \\ 0 & 1 & \dots \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \vdots \\ \ddot{y}_n \end{Bmatrix} + \begin{bmatrix} w_1^2 & 0 & \dots & 0 \\ & w_2^2 & & \\ & & \ddots & \\ & & & w_n^2 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{Bmatrix} = 0$$

$\ddot{y}_i + w_i^2 y_i = 0$

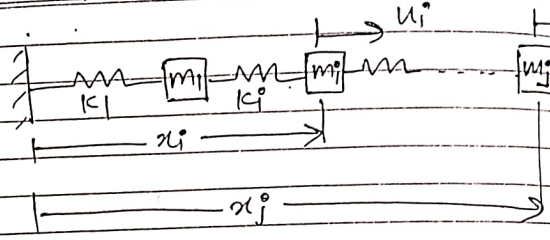
$y_i = A_i e^{i w_i t}$

$y_i = A_i \cos w_i t + B_i \sin w_i t$

Principal coordinates

u_i (i^{th} eigen vectors)

Influence Coefficients



Displacement at unit force

In a complex system, it interest to express the re between the displacement point and the force acti at various points of th This is done by influence coef

Suppose for a moment that just of forces f_j is in action an denote the displacement at a $x = x_i$ due to the force f_j acti $x = x_j$ by u_{ij}.

(displacement at i due to at

Imp:- The flexibility influence coefficie is defined as the deflection $x = x_i$ due to a unit force at $x = x_j$."

$\therefore \; u_{ij} = a_{ij} f_j$

But for a linear system the superposition principle holds true, and total deflection at $x = x_i$, denoted by u_i is given by,

$$u_i = \sum_{j=1}^{n} u_{ij}$$

Thus,

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ \vdots & & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots a_{nn} \end{bmatrix} \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_n \end{Bmatrix}$$

or

$$\{u\} = [a]\{F\} \quad \text{——} (1)$$

Now define the stiffness influence coefficient k_{ij} as the force (F_{ij}) at $x = x_i$ due to a unit displacement at $x = x_j$, the points for which $x \neq x_j$ are fixed

$$F_{ij} = k_{ij} u_j$$

$$F_i = \sum_{j=1}^{n} F_{ij}$$

M-3 Influence coeff Matrix

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ \vdots \\ F_n \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & \cdots \\ k_{21} & k_{22} & \cdots \\ \vdots & & \\ k_{n1} & \cdots & k_{nn} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{Bmatrix}$$

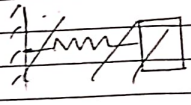
$$\{F\} = [K]\{u\} \quad \text{——} (2)$$

Substituting the value of u from (1), in eqn (2),

$$\{F\} = [K][a]\{F\}$$

$$\Rightarrow [a] = [K]^{-1}$$

Dunkerley's Method :-



Eqⁿ of motion,

$$M\ddot{X} + KX = 0$$

$$[K] = [a]^{-1}$$

$$-\omega^2 MX + [a]^{-1}X = 0$$

$$X = \omega^2 [a][M]X$$

$M\ddot{X} + KX = 0$

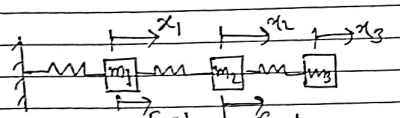
$[K] = [a]^{-1}$

$-\omega^2 MX + [a]^{-1}X = 0$

$X = \omega^2 [a][M]X$

↓

eigen value problem → eigen vector problem



To calculate a_{ij}, we (displacement at i due to unit force at j)

apply force $f_j = 1$ $(j = 1, 2, 3)$ in sequence. In each case, the same unit force is acting everywhere to the left of the point of application $x = x_j$. On the other hand the force is 0 to the right of $x = x_j$.

$$u_{11} = a_{11} = \frac{1}{k_1}$$

$$u_{21} = a_{21} = \frac{1}{k_1}$$

$$u_{31} = a_{31} = \frac{1}{k_1}$$

$$u_{12} = a_{12} = \frac{1}{k_1}$$

$$u_{22} = a_{22} = \frac{1}{k_1} + \frac{1}{k_2}$$

$$u_{32} = a_{32} = \frac{1}{k_1} + \frac{1}{k_2}$$

$$u_{13} = a_{13} = \frac{1}{k_1}$$

$$u_{23} = a_{23} = \frac{1}{k_1} + \frac{1}{k_2}$$

$$u_{33} = a_{33} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}$$

$$[a] = \begin{bmatrix} 1/k_1 & 1/k_1 & 1/k_1 \\ 1/k_1 & 1/k_1 + 1/k_2 & 1/k_1 + 1/k_2 \\ 1/k_1 & 1/k_1 + 1/k_2 & 1/k_1 + 1/k_2 + 1/k_3 \end{bmatrix}$$

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \omega^2 [a] \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

$$x_1 = a_{11} m_1 x_1 \omega^2 + a_{12} m_2 x_2 \omega^2 + a_{13} m_3 x_3 \omega^2$$

$$x_2 = a_{21} m_1 x_1 \omega^2 + a_{22} m_2 x_2 \omega^2 + a_{23} m_3 x_3 \omega^2$$

$$x_3 = a_{31} m_1 x_1 \omega^2 + a_{32} m_2 x_2 \omega^2 + a_{33} m_3 x_3 \omega^2$$

$$\Rightarrow \left(a_{11} m_1 - \frac{1}{\omega^2}\right) x_1 + a_{12} m_2 x_2 + a_{13} m_3 x_3 = 0$$

$$\Rightarrow a_{21} m_1 x_1 + \left(a_{22} m_2 - \frac{1}{\omega^2}\right) x_2 + a_{23} m_3 x_3 = 0$$

$$\Rightarrow a_{31} m_1 x_1 + a_{32} m_2 x_2 + \left(a_{33} m_3 - \frac{1}{\omega^2}\right) x_3 = 0$$

$$\begin{bmatrix} a_{11} m_1 - \frac{1}{\omega^2} & a_{12} m_2 & a_{13} m_3 \\ a_{21} m_1 & a_{22} m_2 - \frac{1}{\omega^2} & a_{23} m_3 \\ a_{31} m_1 & a_{32} m_2 & a_{33} m_3 - \frac{1}{\omega^2} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = 0$$

$$\left(\frac{1}{\omega^2}\right)^3 - \left(a_{11} m_1 + a_{22} m_2 + a_{33} m_3\right)\left(\frac{1}{\omega^2}\right)^2 + \cdots$$

If $\omega_1, \omega_2, \omega_3$ are the roots, then sum of roots

$$\frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} + \frac{1}{\omega_3^2} = a_{11} m_1 + a_{22} m_2 + a_{33} m_3$$

$$\omega_1^2 << \omega_2^2$$
$$\omega_1^2 << \omega_3^2$$

$$\boxed{\frac{1}{\omega_1^2} = a_{11} m_1 + a_{22} m_2 + a_{33} m_3}$$ Dunkerley's eqⁿ

Pg-175, Ex:-4.7

a →

$k_1 = k_2 = k$

$k_3 = 2k$

$m_1 = m_2 = m$

$m_3 = 2m$

$$\frac{1}{\omega_1^2} = \frac{m}{k} + \frac{2m}{k} + (2.5)\frac{2m}{k}$$

$$= \frac{8m}{k}$$

$$\omega_1 = \sqrt{\frac{k}{8m}} = 0.353 \sqrt{\frac{k}{m}}$$

Modal Analysis

$$\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix} = \omega^2 [A][M] \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix}$$

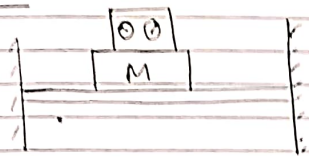
$$\omega^2 \begin{Bmatrix} 6.03 \\ 11.23 \\ 13.81 \end{Bmatrix}$$

we are getting eigen vectors.

Giving $\omega^2 \begin{Bmatrix} 1 \\ (-) \\ (-) \end{Bmatrix}$.

$$\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix} \begin{Bmatrix} 1 \\ -1 \\ -1 \end{Bmatrix}$$

Fig. 10.1, Pg-2.2



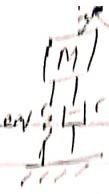
Force and Response

If we vary we don't get mean. Amplitude, phase diff. $N_2 \phi = \omega^2$

ϕ -> How shaker works?

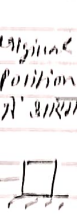
We provide frequency sweep.

$$\text{Tan}\phi = \frac{c\omega_n x}{\phi - \cdot}$$



When we provide a frequency sweep, the frequency of system increases then it reaches the $\omega = \omega_n$ the phase difference is $\pi/2$. Hence, the resonance frequency of the system can be determined.

With the help of shaker we determine the resonance frequency of the system. It will not give natural frequency.

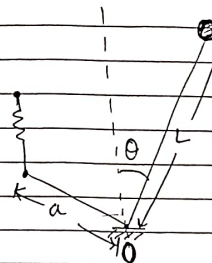


small displacement are difficult

{ small deformations of the structure }

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2.5



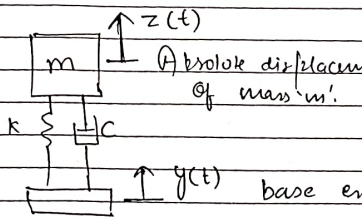
Moment of momentum eq^{n} about O,

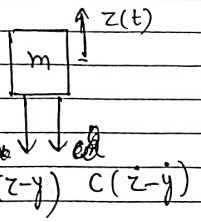
$$I_0\ddot{\theta} = -f_k a + mgL\sin\theta$$

$$f_k = (a\theta - A\cos\omega t)k$$

$$I_0\ddot{\theta} + ka^2\theta - mgL\theta = kA\cos\omega t$$

2.16





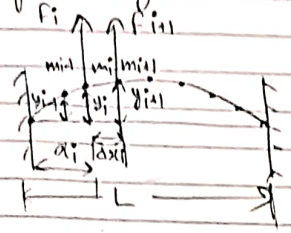
Ans:- $y = A\sin\frac{2\pi}{L}x = A\sin\omega t$

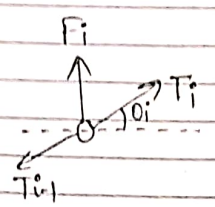
$\omega = \frac{2\pi v}{L}$, $x = vt$

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$$m\ddot{z} = -k(z-y) - c(\dot{z}-\dot{y})$$

Descrete Mathematical Modeling of string Vibration.





$$\tan\theta_{i-1} = \frac{y_i - y_{i-1}}{\Delta x_{i-1}} = \frac{\Delta y_{i-1}}{\Delta x_{i-1}}$$

$$\tan\theta_i = \frac{y_{i+1} - y_i}{\Delta x_i} = \frac{\Delta y_i}{\Delta x_i}$$

$$\Sigma F_y = ma_y$$

$$F_i \cdot 0 + T_i \frac{\Delta y_i}{\Delta x_i} - T_{i-1} \frac{\Delta y_{i-1}}{\Delta x_{i-1}} = m_i \frac{d^2 y_i}{dt^2}$$

$$F_i + \Delta\left(\frac{T_i \Delta y_i}{\Delta x_i}\right) = m_i \frac{d^2 y_i}{dt^2}$$

$$\frac{F_i}{\Delta x_i} + \frac{1}{\Delta x_i}\Delta\left(\frac{T_i \Delta y_i}{\Delta x_i}\right) = \frac{m_i}{\Delta x_i}\frac{d^2 y_i}{dt^2}$$

If $\Delta x_i \to 0$ and x_i is replaced with x

$\frac{F_i}{\Delta x_i} = f(x,t)$ = External force / length

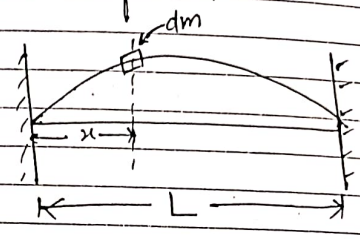
$\frac{m}{\Delta x_i} = \rho(x)$ = mass / length

Hulahoop

$$\rho(x)\frac{\partial^2 y}{\partial t^2} + \frac{\partial}{\partial x}\left(T(x)\frac{\partial y(x,t)}{\partial x}\right) = \rho(x)\frac{\partial^2 y}{\partial t^2}$$

Continuous Mathematical Modeling of string Vibration :→

Consider a string of length 'L' as in fig. The distributed parameters are as follows :-

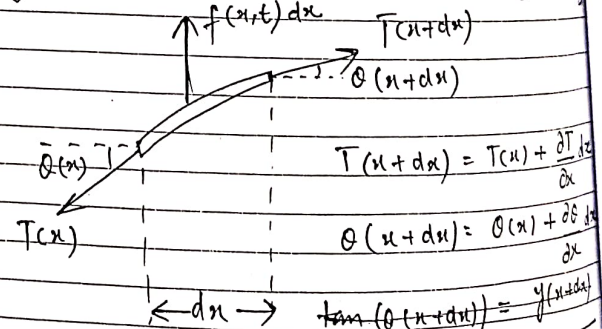


$f(x,t)$ = distributed external force / length

$\rho(x)$ = density as mass / length

$T(x)$ = Tension

$y(x,t)$ = Transverse deflection of string



$T(x+dx) = T(x) + \frac{\partial T}{\partial x}dx$

$\theta(x+dx) = \theta(x) + \frac{\partial \theta}{\partial x}dx$

$\tan(\theta(x+dx)) = y(x+dx)$

$\theta(x,t)$

$f(x,t)dx + T(x+dx)\,\theta(x+dx) - T(x)\,\theta(x) = 0$

$f(x,t)dx + \left[T(x) + \frac{\partial T}{\partial x}dx\right]\left[\theta(x) + \frac{\partial \theta}{\partial x}dx\right] - T(x)\theta(x) = 0$

$f(x,t)dx + T(x)\frac{\partial \theta}{\partial x}dx + \frac{\partial T}{\partial x}dx\,\theta(x) = 0$

$\Rightarrow f(x,t) + T(x)\cdot\frac{\partial \theta}{\partial x} + \frac{\partial T}{\partial x}\theta(x) = 0$

$= \rho(x)\,dx\,\frac{\partial^2 y(x,t)}{\partial t^2}$

$T(x)\frac{\partial \theta}{\partial x} + \frac{\partial T}{\partial x}\theta(x,t)$

$\Rightarrow \frac{\partial}{\partial x}\left(T(x)\,\theta(x,t)\right) + f(x,t) = \rho(x)\frac{\partial^2 y}{\partial t^2}$

derive upto here.

Bending moment dominates	when tension dominates
↓ beam	string
plate	membrane

$Au = \lambda u$ (operates)

$\frac{d}{dx} f(x) = \lambda f(x)$ eigen function.

looking

we are going for synchronous solution :-

$y(x,t) = Y(x)\,F(t)$ ← some function of time.

consider free vibration case :-

$\frac{d}{dx}\left[T(x)\frac{dY(x)}{dx}\right]F(t) = \rho(x)\,Y(x)\frac{d^2F(t)}{dt^2}$

$\frac{1}{\rho(x)Y(x)}\frac{d}{dx}\left(T(x)\frac{dY(x)}{dx}\right) = \frac{1}{F(t)}\frac{d^2F(t)}{dt^2}$

we assume, $\frac{d^2F(t)}{dt^2} = -\omega^2 F(t)$

$\omega^2 = \frac{1}{\rho(x)Y(x)}\frac{d}{dx}\left[T(x)\frac{dY(x)}{dx}\right]$

$f(t) = c\cos(\omega t - \phi)$

Now, boundary conditions are :→

$y(0,t) = 0$

$y(L,t) = 0$

for sake of simplication, $\rho(x) = \rho$

$T(x) = T$

$$\frac{d^2Y(x)}{dx^2} = \frac{-\omega^2 \rho}{T} Y(x)$$

$$\frac{d^2Y(x)}{dx^2} = -\beta^2 Y(x), \quad \text{where } \beta^2 = \frac{\omega^2 \rho}{T}$$

$$Y(x) = A\sin\beta x + B\cos\beta x$$

$Y(0) = 0$

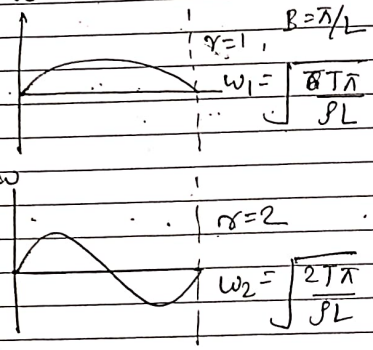
$B = 0$

$Y(L) = 0$

$A\sin\beta L = 0$

$\beta L = r\pi$

$\beta = \frac{r\pi}{L}$



01/11/23

$y(x,t) = Y(x)\,F(t)$ $\qquad \beta = \frac{r\pi}{L}$

$F(t) = C\cos(\omega t - \phi)$ $\qquad \frac{\omega^2\rho}{T} = \beta^2$

$Y_r(x) = A_r \sin\left(\frac{\pi r x}{L}\right)$

$\omega_r = r\pi \sqrt{\frac{T}{\rho L^2}}$

eigen functions are orthogonal

$$\int_0^L f(x)\, Y_r(x)\, Y_s(x)\, dx = 0, \quad \forall\, r \neq s$$

$$= 1, \quad r = s$$

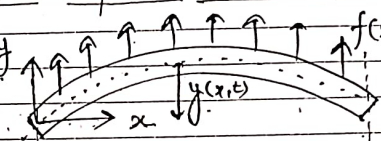
Normalizing the eigen function,

$$\int A_r^2 \int_0^L \sin^2\left(\frac{\pi r x}{L}\right) dx = 1$$

$$\int A_r^2 \frac{L}{2} = 1$$

$$A_r = \sqrt{\frac{2}{\rho L}}$$

Vibration of beam in Bending



Euler Ber b

{ Thicke very Long

We consider the beam in bending

Application of be vibration

where $g(x,t)$ = displacement in transverse direct

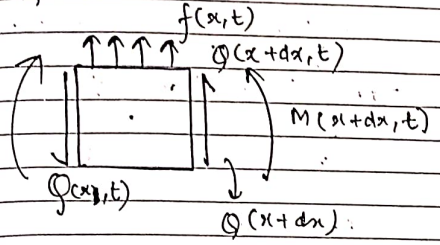
$f(x,t)$ = distributed load / length

$m(x)$ = mass / length

E = Young's Modulus

$I(x)$ = Area moment of inertia about an axis normal to x and y.

FBD of a differential element of length dx at distance x from the left end,



$$-Q(x,t) + Q(x+dx,t) + f(x,t)dx = m(x)dx\frac{\partial^2 y}{\partial t^2}$$

$$Q(x+dx,t) = Q(x,t) + \frac{\partial Q}{\partial x}dx$$

$$\frac{\partial Q}{\partial x} + f(x,t) = m(x)\frac{\partial^2 y}{\partial t^2} \quad \frac{\partial x}{\partial x} \quad \text{——— (1)}$$

(We are dealing with Euler–Bernoulli beam in which ~~rotation~~ is negligible

(contribution of rotation-kinetic energy is negligible compared to translational kinetic energy.)

$$\sum M_o = 0$$

$$-M(x,t) + M(x+dx,t) + Q(x)\frac{dx}{2} + Q(x+dx)\frac{dx}{2} = 0$$

$$\frac{\partial M}{\partial x} + Q(x) = 0$$

$$Q(x) = -\frac{\partial M}{\partial x} \quad \text{——— (A)}$$

Putting in eqn (1),

$$-\frac{\partial^2 M}{\partial x^2} + f(x,t) = m(x)\frac{\partial^2 y}{\partial t^2} \quad \text{——— (2)}$$

The flexural formula is,

$$\boxed{\frac{M}{I} = \frac{E}{R} = \frac{\sigma}{y}}$$

$$\frac{1}{R} = \frac{\partial^2 y}{\partial x^2}$$

$\theta \approx \frac{dy}{dx}$

$$\sigma = \frac{My}{I} = \frac{T}{R}$$

$\sigma = \frac{My}{I} =$

Thus,

$$M = EI(x)\frac{\partial^2 y}{\partial x^2}$$

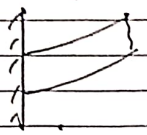
Putting it into (2).

$$-\frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 y}{\partial x^2}\right) + f(x,t) = m(x)\frac{\partial^2 y}{\partial t^2}$$

Differential eqⁿ of motion

It is 2nd order diff. eqⁿ in time, 4th order diff. eqⁿ in space, ∴ we need 4 boundary conditions & 2 initial conditions.

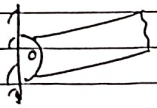
Clamped end



$$y(x)\Big|_{x=0} = 0$$

$$\frac{\partial y}{\partial x}\Big|_{x=0} = 0$$

Pinned end



$$y(x,t)\Big|_{x=0} = 0$$

$$M(x,t)\Big|_{x=0} = 0$$

$$\Rightarrow EI\frac{\partial^2 y}{\partial x^2}\Big|_{x=0} = 0$$



Free end



$$M(x,t)\Big|_{x=0} = 0$$

$$\Rightarrow EI(x)\frac{\partial^2 y}{\partial x^2}\Big|_{x=0} = 0$$

$$Q(x,t)\Big|_{x=0} = 0$$

$$\Rightarrow \frac{\partial^3 y}{\partial x^3}\Big|_{x=0} = 0$$